

PART  
**One**

## **Risk**

**R**isk is about events that we can't foresee. Is there nevertheless some underlying connection between the frequency of past events and the frequency of future events? Between the magnitude of past risks and the magnitude of future risks? Can connections between past and future risks be quantified in some useful way that is not itself risky?

Paradoxically, the risks that are hardest to quantify are the risks of least concern to the institutional investor. The key is the tendency for certain kinds of risks to occur together—i.e., the degree of correlation between the risks. Although uncorrelated risks are the easiest for an institutional investor to diversify, so-called “market” risks, which can't be diversified away, are the easiest to quantify.

J.L.T.

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## CHAPTER 1

## Using Portfolio Composition to Estimate Risk

In recent years a number of financial scholars have commented on the marked degree of co-movement in the prices of securities. Statistical techniques have been applied to measuring the character and degree of co-movement by Donald Farrar, Hester and Feeney, and Benjamin King. Perhaps the best known model of stock prices that recognizes and incorporates the co-movement phenomenon is that of William Sharpe. In Sharpe's model fluctuations in the price of a particular common stock have two causes: (1) fluctuations in the general market level and (2) fluctuations unique to the stock in question. More complicated models than Sharpe's have been proposed and the Sharpe model has occasionally been criticized as being too simple to fit reality (see for example Benjamin King's discussion<sup>1</sup>). Nevertheless, its simplicity gives it great appeal.<sup>2</sup>

We are not the first to apply simple financial models to practical problems involving risk measurement. Marshall Blume tested the applicability of the Sharpe model to the problems of predicting the risk character of simulated rather than actual portfolios.<sup>3</sup> James Fanning, now of Rockefeller Brothers, and Marc Steglitz of Bankers Trust have measured risk in actual common stocks defined in terms of a related, but different, model and applied the results to estimating the risk character of actual portfolios containing these stocks. Although the present paper has benefited substantially from the work of Fanning and Steglitz, in terms of model and approach, we are much closer to Blume than Fanning and Steglitz.

Sometimes it is possible to identify stock price changes with particular news events. Even though the events that cause price changes sometimes seem to be unique, it is nevertheless useful to think of the events that affect prices as drawn at random from a large population, some of which can cause large price changes and some small, many of which have a high degree of uniqueness or individuality, but that, taken as an entire population, have a character that demonstrates some continuity

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This chapter was coauthored with William W. Priest, Jr., Lawrence Fisher, and Catherine A. Higgins.

We are grateful to Marvin Lipson for programming the computer runs, the results of which are reported here. The data for the study were taken from "Price Relative" tapes supplied to us by the Center for Research in Security Prices, the Graduate School of Business, the University of Chicago.

over time. Labor unions will continue to strike; countries will continue to declare war or to make undeclared war; the Fed will continue by turns to tighten up and loosen the money supply; and so forth. Some of these events are felt throughout the economy and have their impact to a greater or lesser degree on the prices of most common stocks. The impact of other events is specific to at most a few companies or industries.

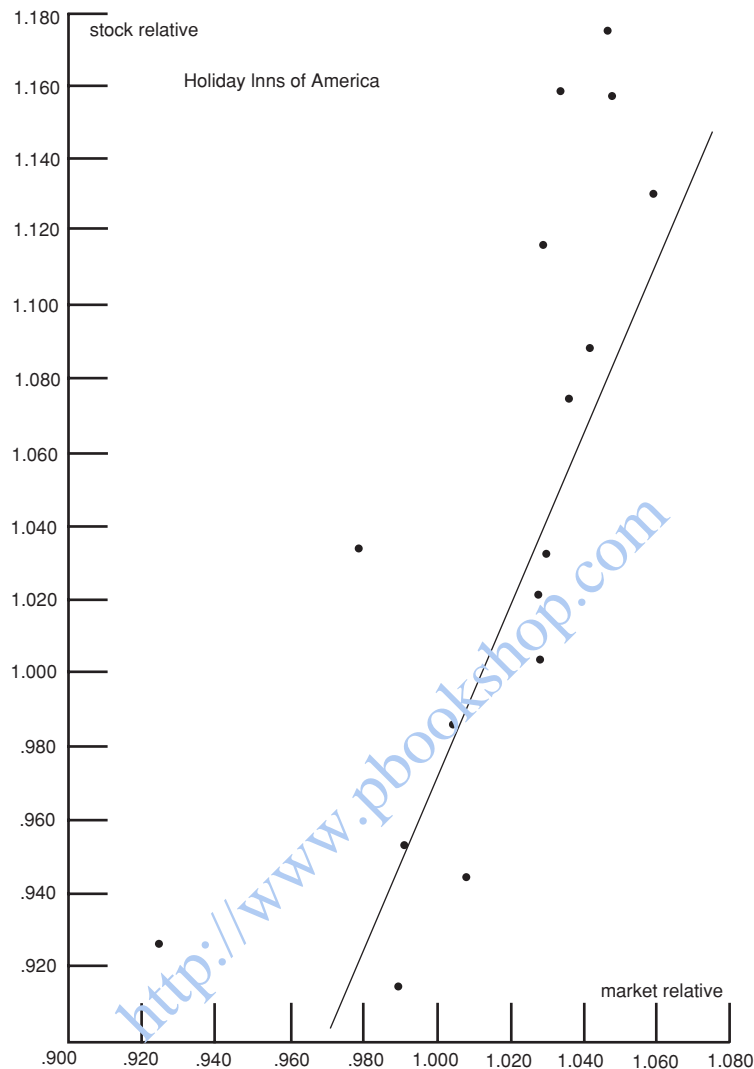
The Sharpe model specifies that price fluctuations in a particular common stock will be the sum of fluctuations due to fluctuations in the market index and fluctuations unique to the stock in question.<sup>4</sup> The risk character of the stock is completely specified under the assumptions of the Sharpe model by specifying two parameters: The first is sensitivity of the stock to market fluctuations. It is common knowledge, however, that price fluctuations in individual common stocks are not completely explained by a market index. We call the portion of price changes left unexplained by a market index the *residual* price changes. The second risk parameter in the Sharpe model is a number that expresses the average magnitude of the residual fluctuations. In Sharpe's model, residual fluctuations are assumed to be independent from one security to another.

Some companies are more sensitive to the impact of events affecting the market index than others. Rapidly growing companies, companies that manufacture capital goods, companies with high fixed costs, and highly levered companies all tend to be more sensitive than companies for which these factors are absent. Companies for which several of these factors are present simultaneously are likely to be particularly sensitive.

The second parameter in the model—the measure of the magnitude of residual fluctuations—tends to be larger for companies in which technological changes in products or processes are taking place very rapidly. It also tends to be larger for one-product companies, companies for which style is an important factor and for companies whose fortunes depend on a single executive. Widely diversified companies and companies with a balanced management team will tend to demonstrate less residual variability than others. A high level of fixed costs or a highly levered capital structure will, of course, amplify specific risk in the same way that it amplifies market risk.

Exhibit 1.1 demonstrates the meaning of the risk parameters for individual stocks in graphical terms. The horizontal axis measures the change in a market index (Fisher's Combination Investment Performance Index<sup>5</sup>). The vertical axis measures the change in the value of the security in question. Both are measured as the ratio of value at the end of a month (including intervening dividends) to value at the beginning. A straight line has been fitted to the data points in Exhibit 1.1. The slope of the line is a measure of the sensitivity of the value of the security to fluctuations in the market index. The spread of data points around the line of best fit is a measure of residual variability.

The important distinction between market variability and residual variability in the individual security is that they affect portfolio returns in different ways. Sensitivity of a portfolio to variations in the market index is the average of the sensitivities of the individual securities held, weighted by the amounts. Residual variability in individual securities, on the other hand, tends to combine in such a way that it looms relatively less important in a portfolio (in comparison with market variability) than in the securities which comprise it. The spread is measured in terms of a number statisticians call the *residual variance*, which in terms of the present application is



**EXHIBIT 1.1** Price Relatives for a Common Stock Versus Price Relatives for “The Market”

an average of the squares of the (residual) fluctuations over the time period covered by the sample. Under the assumptions of the Sharpe model, there is a very simple rule for determining how specific risk in individual securities combines to determine specific risk in the portfolio: The residual variance for the portfolio can be expressed in terms of residual variance  $\sigma_i^2$  for the individual stocks. Letting  $x_i$  equal the number of shares of the  $i$ th stock held in the portfolio and assuming that the relevant measures are expressed on a per-share basis, we have

$$\text{Residual variance} = \sum x_i^2 \sigma_i^2 \quad (1.1)$$

In like fashion we can define *market variance* as the average over time of (squared) fluctuations in portfolio value due to market fluctuations. Letting regression-slope coefficients  $B_i$  measure sensitivity of prices of individual stocks to fluctuations in the general market, and a variance  $\sigma_m^2$  describe the variability of the general market, we have

$$\text{Market variance} = [\sum x_i B_i]^2 \sigma_m^2 \quad (1.2)$$

Finally, expressions (1) and (2) can be combined to estimate total portfolio variance  $\sigma^2$ . We have

$$\sigma^2 = [\sum x_i B_i]^2 \sigma_m^2 + \sum x_i^2 \sigma_i^2 \quad (1.3)$$

Equation 1.3 will hold only approximately since the residual variances for individual stocks are not strictly independent. As previously noted, a number of writers have challenged the Sharpe model on the assumptions underlying the way specific risks in individual securities combine in a portfolio. Nevertheless Equation 1.3 is probably

**EXHIBIT 1.2** Exponentially-Smoothed Estimates of Market Volatility ( $B$ ) and Residual Variance ( $\sigma^2$ )

Holiday Inns of America, Inc.		
Month	$B$	$\sigma^2$
3/64	0.653	0.00643
4/64	4.604	0.01051
5/64	4.307	0.00880
6/64	4.300	0.00754
7/64	4.314	0.00656
8/64	3.797	0.00604
9/64	3.132	0.00571
10/64	2.725	0.00528
11/64	3.025	0.00670
12/64	2.812	0.00635
1/65	2.923	0.00589
2/65	3.073	0.00548
3/65	2.961	0.00520
4/65	2.938	0.00489
5/65	3.006	0.00461
6/65	2.943	0.00436
7/65	1.742	0.00505
8/65	1.746	0.00479
9/65	2.036	0.00535
10/65	2.194	0.00581
11/65	2.356	0.00579
12/65	2.423	0.00572
1/66	2.391	0.00551
2/66	2.410	0.00529
3/66	2.427	0.00527

**EXHIBIT 1.3** How Market Risk Varies: Examples from the U.S. Market

Ranked by $B_i$		
1.	Holiday Inns	2.427
2.	Warner & Swasey	2.088
3.	Admiral	2.025
4.	Collins Radio	1.954
5.	General Instru.	1.898
6.	Vornado	1.817
7.	Piper Aircraft	1.798
8.	Beckman Instru.	1.763
9.	Fairchild Camera	1.717
10.	Northwest Airlines	1.687
11.	Commonwealth Oil	1.679
12.	Max Factor	1.624
13.	Phila & Reading	1.609
14.	Raytheon	1.595
15.	Bell & Howell	1.595
16.	Avon	1.589
17.	Texas Instru.	1.575
18.	TWA	1.524
19.	Pan Am World Airways	1.517
20.	Financial Federation	1.506
21.	Ampex	1.505
22.	First Charter Financial	1.504
23.	Control Data	1.500
24.	Crowell Collier	1.498
25.	Foxboro	1.496
26.	Universal Oil Prod.	1.479
27.	EJ Korvette	1.472
28.	William H. Rorer	1.446
29.	Magnavox	1.438
30.	Polaroid	1.425
31.	Eastern Airlines	1.404
32.	Cerro	1.398
33.	Reynolds Metal	1.394
34.	National Airlines	1.378
35.	Perkin Elmer	1.378
36.	Zenith Radio	1.358
37.	Cons. Electronics Inds.	1.375
38.	Rayonier	1.449
39.	Celanese	1.357
40.	Xerox	1.352
41.	Revere Copper & Brass	1.350
42.	Motorola	1.344
43.	Litton Inds.	1.337
44.	Penn RR	1.332
45.	Ginn & Co.	1.320
46.	Douglas Aircraft	1.308
47.	Indian Head Mills	1.307
48.	Mallory	1.306
49.	Sunstrand	1.303
50.	Delta Airlines	1.302
51.	I T & T	1.296
52.	Carter Products	1.291
53.	General Precision	1.290
54.	Sperry-Rand	1.290
55.	Western Airlines	1.277
56.	Hewlett Packard	1.262
57.	ACF Inds.	1.242
58.	Great Northern Paper	1.233
59.	Owens Corning	1.207
60.	Kayser Roth	1.205
61.	Grace	1.194
62.	Kaiser Alum.	1.191
63.	Frito Lay	1.17
64.	Air Prd. & Chem.	1.157
65.	Allis Chalmers	1.157
66.	Whirlpool	1.156
67.	Pfizer Chas.	1.133
68.	Corning Glass Works	1.123
69.	General Dynamics	1.120
70.	Biegelow Sanford	1.115
71.	Texas Oil & Gas	1.114
72.	Wetson & Co.	1.105
73.	Pennzoil	1.098
74.	National Can	1.093
75.	Schering	1.092
76.	Union Bag Camp Paper	1.078
77.	Burroughs	1.072
78.	Mueller Brass	1.059
79.	Allied Supermarkets	1.054
80.	MGM	1.048
81.	Bethlehem Steel	1.045
82.	Fibreboard Paper Prds.	1.039
83.	Olin Mathieson	1.035
84.	Harbison Walker	1.030
85.	Chesebrough Ponds	1.022
86.	Southern Co.	1.019
87.	Aluminum Co Amer.	1.011
88.	W. Virginia Pulp & Paper	1.000
89.	Caterpillar Tractor	0.985
90.	Beaunit	0.982
91.	Crown Cork & Seal	0.978
92.	Tidewater	0.978
93.	Chrysler	0.966
94.	Montgomery Ward	0.953
95.	Texas Gulf Sulphur	0.945
96.	Cons. Cigar	0.936
97.	Ex-Cell-O	0.936
98.	Westinghouse Electric	0.932
99.	Upjohn	0.920

**EXHIBIT 1.3** (Continued)

100.	Holt Rinehart & Winston	0.914	117.	Marathon	0.752
101.	Wesco Financial	0.911	118.	Allied Chemical	0.738
102.	Grumman	0.904	119.	Gulf Oil	0.727
103.	Halliburton	0.893	120.	McDonnell Aircraft	0.721
104.	Florida Pwr. & Light.	0.888	121.	Socony Mobil Oil	0.665
105.	Cone Mills	0.887	122.	Texaco	0.661
106.	Colgate Palmolive	0.851	123.	Monsanto	0.649
107.	Union Oil Cal.	0.850	124.	Sunbeam	0.639
108.	Merck	0.850	125.	S. Carolina Elec. & Gas	0.610
109.	Columbia Brdestg.	0.837	126.	Central Southwest	0.607
110.	Bobbie Brooks	0.825	127.	Abbott Labs	0.593
111.	Gillette	0.819	128.	Standard Oil Cal.	0.587
112.	Lockheed	0.805	129.	Petrolane Gas Service	0.576
113.	United Fruit	0.782	130.	Beneficial Finance	0.565
114.	Coastal States Gas Prod.	0.774	131.	Gulf States Utilities	0.549
115.	United Carr	0.764	132.	Southwestern Public Sve.	0.451
116.	IBM	0.757	133.	AT&T	0.403

the simplest model which has any reasonable hope of predicting the risk character of a diversified portfolio.

The values for the regression coefficients and residual variances are obtained by regressing price change histories for individual common stocks against a suitable market average. From these values and composition data, a model of the risk character of the portfolio is constructed. Predictions of a change in the value of the portfolio (given the change in market level) are compared with the actual change in order to test this model.

Our basic idea (in which we were anticipated by the work of Fanning and Steglitz, and also by that of Marshall Blume) is that if we knew the risk parameters for individual common stocks then we could estimate the risk character of a portfolio instantaneously—even though the composition was continuously changing. In order to test this idea we have studied all the common stocks held in an actual mutual-fund portfolio during a period of more than two years. The price history of each common stock was traced back as far as conveniently possible—in some cases, up to 40 years. From the price and dividend histories for the common stocks, we made running estimates of the risk parameters for each common stock held. Then, once a month for each month during the test period, we made an instantaneous estimate of the risk character of the mutual fund portfolio, based on its composition at the end of that month. Our estimate of *market* risk for the portfolio enabled us to predict how rapidly the value of the fund would change as the market level fluctuated. Our estimate of *residual* risk for the fund gives an estimate of the amount by which the true market value of the fund will differ from our predictions. For each month of the test period we estimated both risk parameters for the fund and observed the actual change in market level and the actual change in the value of the fund. How well we succeeded in predicting the observed changes in the value of the fund, given the actual changes in market level, is discussed at the end of this paper.

**EXHIBIT 1.4** How Specific Risk Varies: Examples from the U.S. Market

Ranked by Residual Variance					
1.	Control Data Corp.	0.02116	49.	Ginn & Co.	0.00628
2.	Fairchild Camera & Instru.	0.02071	50.	Kaiser Alum. & Chem.	0.00622
3.	Texas Gulf Sulphur	0.01576	51.	Pennzoil Co.	0.00619
4.	Texas Instru.	0.01453	52.	Beaunit Corp.	0.00613
5.	Admiral Corp.	0.01431	53.	Kayser Roth Corp.	0.00603
6.	EJ Korvette	0.01344	54.	Lockheed Aircraft	0.00603
7.	Collins Radio	0.01326	55.	Litton Inds.	0.00590
8.	Wesco Financial Corp.	0.01244	56.	Grumman Aircraft	0.00582
9.	Xerox Corp.	0.01217	57.	Western Airlines	0.00578
10.	Financial Federation	0.01200	58.	Bobbie Brooks	0.00576
11.	General Instru. Corp.	0.01145	59.	Sunstrand Corp.	0.00573
12.	First Charter Finan. Corp.	0.01045	60.	Motorola Inc.	0.00571
13.	Ampex Corp.	0.01011	61.	United Fruit	0.00565
14.	Cons. Electronics Inds.	0.00946	62.	Great Northern Paper Co.	0.00564
15.	Beckman Instru.	0.00939	63.	Schering Corp.	0.00558
16.	Commonwealth Oil Refining	0.00900	64.	Sperry-Rand Corp.	0.00540
17.	Raytheon Co.	0.00893	65.	Wetson & Co.	0.00540
18.	Vornado Inc.	0.00892	66.	Cerro Corp.	0.00539
19.	Max Factor & Co.	0.00878	67.	Southern Co.	0.00537
20.	Polaroid Corp.	0.00874	68.	Chrysler Corp.	0.00532
21.	Magnavox Co.	0.00861	69.	Cons. Cigar Corp.	0.00529
22.	Crowell-Collier	0.00820	70.	Holiday Inns of Amer.	0.00527
23.	William H. Rorer	0.00812	71.	Tidewater Oil Co.	0.00515
24.	Douglas Aircraft Co.	0.00780	72.	Bigelow-Sanford Inc.	0.00504
25.	Zenith Radio Corp.	0.00765	73.	Air Prod. & Chem.	0.00501
26.	Texas Oil & Gas	0.00762	74.	Indian Head Mills	0.00498
27.	General Dynamics	0.00752	75.	Burroughs Corp.	0.00493
28.	McDonnell Aircraft Corp.	0.00730	76.	Warner & Swasey Co.	0.00489
29.	TWA Inc.	0.00725	77.	Holt Rinehart & Winston	0.00488
30.	Bell & Howell Co.	0.00717	78.	Revere Copper & Brass	0.00482
31.	Hewlett Packard Co.	0.00711	79.	MGM	0.00480
32.	Crown Cork & Seal	0.00705	80.	Whirlpool Corp.	0.00476
33.	Piper Aircraft Corp.	0.00697	81.	Celanese Corp Amer.	0.00475
34.	Universal Oil Prds.	0.00694	82.	Owens Corning Fiberglass	0.00465
35.	Northwest Airlines	0.00692	83.	Corning Glass Works	0.00449
36.	Perkin Elmer Corp.	0.00691	84.	Sunbeam Corp.	0.00447
37.	Carter Prod. Inc.	0.00688	85.	Rayonier Inc.	0.00439
38.	Mueller Brass Co.	0.00686	86.	Alum. Co. Amer.	0.00433
39.	Eastern Airlines	0.00685	87.	Petrolane Gas Srv. Inc.	0.00429
40.	Reynolds Metals	0.00679	88.	Intern. T & T	0.00428
41.	National Can Corp.	0.00668	89.	Gillette Co.	0.00406
42.	Foxboro Co.	0.00666	90.	ACF Inds.	0.00404
43.	Pan Am Wld Airways	0.00666	91.	Penn. RR	0.00404
44.	Phil. & Reading Corp.	0.00660	92.	Columbia Brdestg.	0.00402
45.	National Airlines	0.00659	93.	W. Virginia Pulp & Paper	0.00402
46.	Allied Supermarkets	0.00648	94.	Avon Prod.	0.00401
47.	Delta Air Lines Inc.	0.00645	95.	Frito Lay	0.00396
48.	General Precision	0.00636	96.	Fibreboard Paper Prds.	0.00394
			97.	Merck & Co.	0.00387
			98.	Mallory Pr. & Co	0.00373

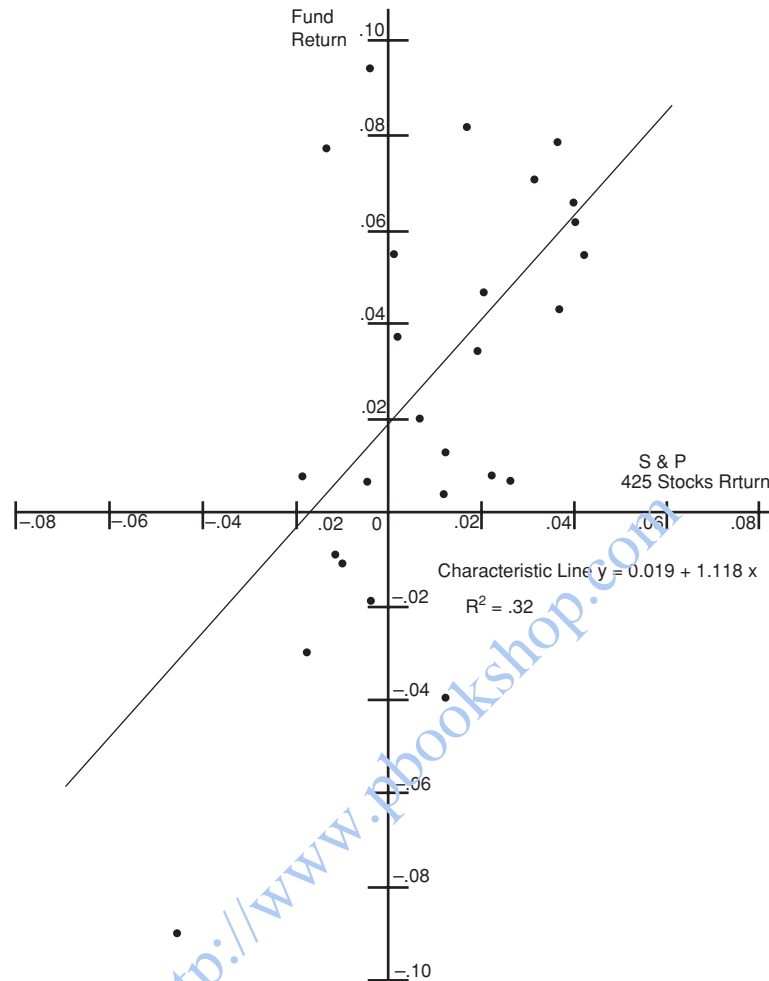
**EXHIBIT 1.4** (Continued)

99.	Caterpillar Tractor	0.00363	117.	Olin Mathieson Chem.	0.00274
100.	United Carr Inc.	0.00359	118.	Gulf Oil	0.00267
101.	Bethlehem Steel	0.00356	119.	Westinghouse Electric	0.00267
102.	Abbott Lab	0.00354	120.	Grace WR & Co.	0.00255
103.	Upjohn Co.	0.00353	121.	Union Oil of Cal.	0.00245
104.	Coastal States Gas Prd. Co.	0.00352	122.	IBM	0.00242
105.	Chesebrough Ponds Inc.	0.00341	123.	Florida Pwr. & Light	0.00234
106.	Ex-Cell-O	0.00340	124.	Beneficial Finance	0.00232
107.	Cone Mills	0.00336	125.	Socony Mobil Oil	0.00225
108.	Colgate Palmolive	0.00330	126.	Texaco	0.00217
109.	Halliburton Co.	0.00331	127.	Gulf States Utilities	0.00216
110.	Monsanto Co.	0.00329	128.	S. Carolina Electric & Gas	0.00203
111.	Allis Chalmers Mfg.	0.00327	129.	Central S. West	0.00200
112.	Marathon Oil Co.	0.00323	130.	Allied Chem. Corp.	0.00185
113.	Union Bag Camp Paper	0.00318	131.	Southwestern Public Svc.	0.00181
114.	Montgomery Ward	0.00309	132.	Standard Oil of Cal.	0.00174
115.	Pfizer Chas	0.00294	133.	AT&T	0.00091
116.	Harbison Walker Refractories	0.00291			

Although the risk character of a fund may change quickly if the composition of the fund changes, our scheme assumes that risk parameters for individual common stocks change relatively slowly. In most cases the assumption seems valid to us, since they change through the gradual evolution of products, manufacturing processes, and markets. The risk character of a company's common stock may change quickly, however, if the company enters into a wide-ranging diversification program or undergoes a profound change in capital structure. The essence of the measurement problem is that they are measured subject to random fluctuations in the data and that reliable estimates can be obtained only with samples large enough to "average out" these random fluctuations to some degree. Unfortunately, over time, the underlying parameters that we are attempting to measure are themselves changing. Thus we are confronted with an inescapable dilemma: If we confine our samples to very recent data, possible error due to random fluctuations in sample data may be excessively large. If, on the other hand, we include in our sample a longer time span, we may be including data that are no longer relevant because of changes over time in the risk character of the common stock in question. In principle, there are ways of weighting more and less recent data that are optimal in the sense of minimizing the combined effects of both problems. We are currently experimenting with techniques that select optimum weights automatically. For this study, however, we used exponential smoothing techniques with arbitrary weights (see Appendix 1.1).

Using exponential smoothing we obtained running (that is, continually changing) estimates of the risk character of a large group of common stocks covering as much as 40 years. Exhibit 1.2 shows how estimates of the risk parameters for a single common stock have behaved over time.

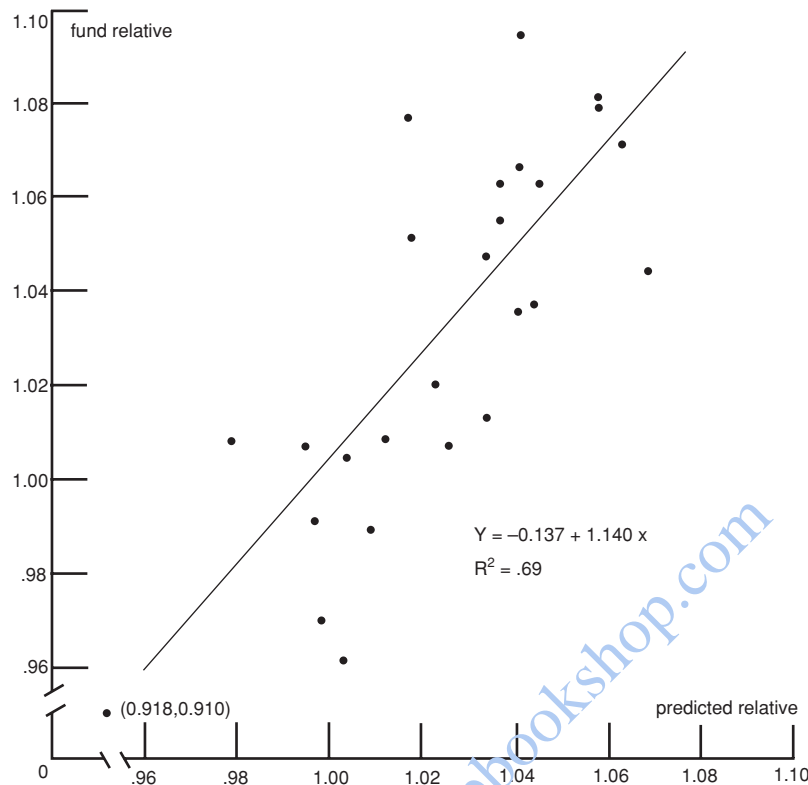
Perhaps the most striking thing about our estimates of risk parameters for individual common stocks is the range of values encountered in our modest sample. Exhibit



**EXHIBIT 1.5** Fund Return versus Market Return

1.3 shows current estimates for market risk of stocks in the sample. A regression-slope coefficient  $B_i$  equal to 1 implies that the stock in question has an average degree of market risk, and that if the market rises or falls 10 percent other things equal the stock in question will rise or fall respectively 10 percent. Values  $B_i$  in Exhibit 1.3 range from less than  $\frac{1}{2}$  to more than  $2\frac{1}{2}$ . In other words some stocks in Exhibit 1.3 have more than 5 times as much market risk as some others. Clearly the degree of market risk in a portfolio is determined, not only by the proportions devoted to common stock and fixed income securities respectively, but also by the kind of common stocks held.

Exhibit 1.4 shows current estimates of  $\sigma_1^2$ , the spread in residual risk for the common stocks studied. Here too the range is impressive. As one might expect there is some tendency for stocks that rank high in Exhibit 1.3 to rank high in Exhibit 1.4.



**EXHIBIT 1.6** Fund Return versus Predictions Based on Risk Character of Stocks Listed

Ultimately our interest in the risk character of common stocks derives entirely from their possible impact on the risk character of a portfolio. The risk character of the mutual fund portfolio considered in this study<sup>6</sup> is displayed in Exhibit 1.5. The rate of return for an appropriate market index (the same Fisher Index referred to above) is measured along the horizontal axis and a rate of return for the fund is measured on the vertical axis. (The scatter diagram in Exhibit 1.5 covers 27 consecutive months of investment results for the fund.) The slope of the regression line fitted to these points (the Characteristic Line) is a measure of the average level of market risk in the fund over this period. If the actual level of market risk in the fund had been maintained constant over the period, then the dispersion of month-to-month results around the line of best fit would be a measure of the degree of specific risk in the fund. If on the other hand, the actual degree of market risk in the fund was changing from month to month, then the dispersion of the data around the line of best fit overstates the degree of specific risk in the fund. It is obviously necessary to accumulate data over a substantial period of time in order to measure market risk in a fund using the Characteristic Line technique.

Exhibit 1.6 compares actual month-to-month results for the fund with results predicted, using the technique described in this paper. A comparison of Exhibit 1.5

and 1.6 shows that our forecast of investment results for the fund is improved by using risk estimates for the individual common stocks held. In fact, roughly half the variance left unexplained by the Characteristic Line is accounted for by allowing for changes in the composition of the fund (hence changes in the risk character of the fund) over the sample period. We conclude that our risk-measuring technique is producing numbers that are both meaningful and useful—not only for estimating portfolio risk after the fact but also for estimating the impact on fund risk of making contemplated changes in the composition of the fund.

### APPENDIX 1.1

The following formulas indicate schematically how we used Exponential Smoothing to get continuously-updated estimates of risk parameters for individual common stocks. Let  $x_i(t)$  be the rate of return for the  $i$ th stock in period  $t$  and define  $\bar{x}_i(t)$  implicitly as our estimate of the current expected value around which  $x(t)$  is fluctuating. Let  $\mu(t)$  be the rate of return for an appropriate market index and define  $\bar{\mu}(t)$  analogously to  $\bar{x}(t)$ . Then define the covariance matrix  $\sigma_i^2$  by

$$\sigma_i^2 = \begin{pmatrix} \sigma_{ii}^2 & \sigma_{i\mu}^2 \\ \sigma_{i\mu}^2 & \sigma_{\mu\mu}^2 \end{pmatrix} = \begin{pmatrix} (x_i - \bar{x}_i)^2 & (x_i - \bar{x}_i)(\mu - \bar{\mu}) \\ (x_i - \bar{x}_i)(\mu - \bar{\mu}) & (\mu - \bar{\mu})^2 \end{pmatrix} \quad (1.4)$$

and define  $\sigma_i^2(t)$  implicitly by the relation

$$\sigma_i^2(t) = \alpha \sigma_i^2(t) + (1 - \alpha) \bar{\sigma}_i^2(t - 1)$$

where  $\alpha$  is the same smoothing constant as before. Then our estimates of the regression parameters  $\beta_i(t)$  and  $\sigma_i^2(t)$  are given by

$$\beta_i(t) = \frac{\sigma_{i\mu}^2(t)}{\sigma_{\mu\mu}^2(t)} \quad (1.5)$$

$$\sigma_i^2(t) = \sigma_{ii}^2(t) - \beta_i^2(t) \sigma_{\mu\mu}^2(t) \quad (1.6)$$

It can be seen from these formulas that, when the true values of  $\beta_i$  and  $\sigma_i^2$  are changing along a steady trend, our estimates will tend to lag somewhat behind the true values.

### Notes

1. "Market and Industry Factors in Stock Price Behavior," *Journal of Business*, Volume 39, Number 1, Part II ("Supplement," January, 1966), pp. 139–190.
2. It should be noted that in conversation with one of us (Fisher) in 1964 or 1965, Harry Markowitz expressed the opinion that the degree of clustering of fluctuations found by

- King would not cause portfolios to show riskiness substantially different from that estimated using the Sharpe model.
3. Unpublished monograph *The Empirical Adequacy of Portfolio Theory*, submitted to *Journal of Business*, July, 1968.
  4. Sharpe's model was nearly anticipated by M. F. M. Osborne in his celebrated paper "Brownian Motion in the Stock Market," (*Operations Research*, Vol. 7, March–April, 1959). Osborne considered an "ensemble consisting of 1,000 pennies and one gold piece." The outcome of the toss of the gold piece affected the prices of 1,000 stocks; the effect of the outcome of tossing each of the 1,000 pennies was unique to a single stock.
  5. Described in "Some New Stock Market Indexes," *Journal of Business*, *loc. cit.*, pp. 191–225.
  6. The "fund" studied was the portion of Diversified Growth Stock Fund that was invested in common stocks listed on the New York Stock Exchange. These stocks comprised over 90 per cent of the net assets of the fund throughout the period studied (December 31, 1964 to March 31, 1966).

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