CHAPTER 1

Money Market Interest Rates

An interest rate is a summary statistic about the cash flows on a debt security such as a loan or a bond. As a statistic, it is a number that we calculate. An objective of this chapter is to demonstrate that there are many ways to do this calculation. Like many statistics, an interest rate can be deceiving and misleading. Nevertheless, we need interest rates to make financial decisions about borrowing and lending money and about buying and selling securities. To avoid being deceived or misled, we need to understand how interest rates are calculated.

It is useful to divide the world of debt securities into *short-term money markets* and *long-term bond markets*. The former is the home of money market instruments such as Treasury bills, commercial paper, bankers acceptances, bank certificates of deposit, and overnight and term sale-repurchase agreements (called "repos"). The latter is where we find coupon-bearing notes and bonds that are issued by the Treasury, corporations, federal agencies, and municipalities. The key reference interest rate in the U.S. money market is 3-month LIBOR (the London inter-bank offered rate); the benchmark bond yield is on 10-year Treasuries.

This chapter is on money market interest rates. Although the money market usually is defined as securities maturing in one year or less, much of the activity is in short-term instruments, from overnight out to six months. The typical motivation for both issuers and investors is cash management arising from the mismatch in the timing of revenues and expenses. Therefore, primary investor concerns are liquidity and safety. The instruments themselves are straightforward and entail just two cash flows, the purchase price and a known redemption amount at maturity.

Let's start with a practical money market investment problem. A fund manager has about $1 million to invest and needs to choose between two 6-month securities: (1) commercial paper (CP) quoted at 3.80% and (2) a bank certificate of deposit (CD) quoted at 3.90%. Assuming that the credit risks are the same and any differences in liquidity and taxation are immaterial, which investment offers the better rate of return, the CP at 3.80% or the CD at 3.90%? To the uninitiated, this must seem like a trick question—surely, 3.90% is higher than 3.80%. If we are correct in our assessment that the risks are the same, the CD appears to pick up an extra 10 basis points. The initiated know that first it is time for a bit of bond math.

Interest Rates in Textbook Theory

You probably were first introduced to the time value of money in college or in a job training program using equations such as these:

$$FV = PV * (1+i)^N \text{ and } PV = \frac{FV}{(1+i)^N} \qquad (1.1)$$

where FV = future value, PV = present value, i = interest rate per time period, and N = number of time periods to maturity.

The two equations are the same, of course, and merely are rearranged algebraically. The future value is the present value moved forward along a time trajectory representing compound interest over the N periods; the present value is the future value discounted back to day zero at rate i per period.

In your studies, you no doubt worked through many time-value-of-money problems, such as: How much will you accumulate after 20 years if you invest $1,000 today at an annual interest rate of 5%? How much do you need to invest today to accumulate $10,000 in 30 years assuming a rate of 6%? You likely used the time-value-of-money keys on a financial calculator, but you just as easily could have plugged the numbers into the equations in 1.1 and solved via the arithmetic functions.

$$\$1{,}000 * (1.05)^{20} = \$2{,}653 \text{ and } \frac{\$10{,}000}{(1.06)^{30}} = \$1{,}741$$

The interest rate in standard textbook theory is well defined. It is the *growth rate of money* over time—it describes the trajectory that allows $1,000 to grow to \$2,653 over 20 years. You can interpret an interest rate as an *exchange rate across time*. Usually we think of an exchange rate as a trade between two currencies (e.g., a spot or a forward foreign exchange rate between the U.S. dollar and the euro). An interest rate tells you the amounts in the same currency that you would accept at different points in time. You would be indifferent between \$1,741 now and \$10,000 in 30 years, assuming that 6% is the correct exchange rate for you. An interest rate also indicates the *price of money*. If you want or need \$1,000 today, you have to pay 5% annually to get it, assuming you will make repayment in 20 years.

Despite the purity of an interest rate in time-value-of-money analysis, you cannot use the equations in 1.1 to do interest rate and cash flow calculations on money market securities. This is important: *Money market interest rate calculations do not use textbook time-value-of-money equations*. For a money manager who has \$1,000,000 to invest in a bank CD paying 3.90% for half of a year, it is *wrong* to calculate the future value in this manner:

$$\$1,000,000 * (1.0390)^{0.5} = \$1,019,313$$

While it is tempting to use $N = 0.5$ in equation 1.1 for a 6-month CD, it is not the way money market instruments work in the real world.

Money Market Add-on Rates

There are two distinct ways that money market rates are quoted: as an *add-on rate* and as a *discount rate*. Add-on rates generally are used on commercial bank loans and deposits, including certificates of deposit, repos, and fed funds transactions. Importantly, LIBOR is quoted on an add-on rate basis. Discount rates in the U.S. are used with T-bills, commercial paper, and bankers acceptances. However, there are no hard-and-fast rules regarding rate quotation in domestic or international markets. For example, when commercial paper is issued in the Euromarkets, rates typically are on an add-on basis, not a discount rate basis. The Federal Reserve lends money to commercial banks at its official "discount rate." That interest rate, however, actually is quoted as an add-on rate, not as a discount rate. Money market rates can be confusing—when in doubt, verify!

First, let's consider rate quotation on a bank certificate of deposit. Add-on rates are logical and follow *simple interest* calculations. The interest is added on to the principal amount to get the redemption payment at maturity. Let *AOR* stand for add-on rate, *PV* the present value (the initial principal amount), *FV* the future value (the redemption payment including interest), *Days* the number of days until maturity, and *Year* the number of days in the year. The relationship between these variables is:

$$FV = PV + \left[PV * AOR * \frac{Days}{Year}\right] \tag{1.2}$$

The term in brackets is the interest earned on the bank CD—it is just the initial principal times the annual add-on rate times the fraction of the year.

The expression in 1.2 can be written more succinctly as:

$$FV = PV * \left[1 + \left(AOR * \frac{Days}{Year}\right)\right] \tag{1.3}$$

Now we can calculate accurately the future value, or the redemption amount including interest, on the $1,000,000 bank CD paying 3.90% for six months. But first we have to deal with the fraction of the year. Most money market instruments in the U.S. use an "actual/360" day-count convention. That means *Days*, the numerator, is the actual number of days between the settlement date when the CD is purchased and the date it matures. The denominator usually is 360 days in the U.S. but in many other countries a more realistic 365-day year is used. Assuming that *Days* is 180 and *Year* is 360, the future value of the *CD* is $1,019,500, and not $1,019,313 as incorrectly calculated using the standard time-value-of-money formulation.

$$FV = \$1{,}000{,}000 * \left[1 + \left(0.0390 * \frac{180}{360}\right)\right] = \$1{,}019{,}500$$

Once the bank CD is issued, the *FV* is a known, fixed amount. Suppose that two months go by and the investor—for example, a money market mutual fund—decides to sell. A securities dealer at that time quotes a bid rate of 3.72% and an asked (or offered) rate of 3.70% on 4-month CDs corresponding to the credit risk of the issuing bank. Note that securities in the money market trade on a *rate basis*. The bid rate is higher than the ask rate so that the security will be bought by the dealer at a lower price than it is sold. In the bond market, securities usually trade on a *price basis*.

The sale price of the CD after the two months have gone by is found by substituting $FV = \$1{,}019{,}500$, $AOR = 0.0372$, and $Days = 120$ into equation 1.3.

$$\$1{,}019{,}500 = PV * \left[1 + \left(0.0372 * \frac{120}{360}\right)\right], \quad PV = \$1{,}007{,}013$$

Note that the dealer buys the CD from the mutual fund at its quoted bid rate. We assume here that there are actually 120 days between the settlement date for the transaction and the maturity date. In most markets, there is a one-day difference between the trade date and the settlement date (i.e., next-day settlement, or "T + 1").

The general pricing equation for add-on rate instruments shown in 1.3 can be rearranged algebraically to isolate the *AOR* term.

$$AOR = \left(\frac{Year}{Days}\right) * \left(\frac{FV - PV}{PV}\right) \tag{1.4}$$

This indicates that a money market add-on rate is an annual percentage rate (APR) in that it is the number of time periods in the year, the first term in parentheses, times the interest rate per period, the second term. $FV - PV$ is the interest earned; that divided by amount invested *PV* is the rate of return on the transaction for that time period. To annualize the periodic rate of return, we simply multiply by the number of periods in the year (*Year*/*Days*). I call this the *periodicity* of the interest rate. If *Year* is assumed to be 360 days and *Days* is 90, the periodicity is 4; if *Days* is 180, the periodicity is 2. Knowing the periodicity is critical to understanding an interest rate.

APRs are widely used in both money markets and bond markets. For example, the typical fixed-income bond makes semiannual coupon payments. If the payment is \$3 per \$100 in par value on May 15^{th} and November 15^{th} of each year, the coupon rate is stated to be 6%. Using an APR in the money market does require a subtle yet important assumption, however. It is assumed implicitly that the transaction can be replicated at the same rate per period. The 6-month bank CD in the example can have its *AOR* written like this:

$$AOR = \left(\frac{360}{180}\right) * \left(\frac{\$1{,}019{,}500 - \$1{,}000{,}000}{\$1{,}000{,}000}\right) = 0.0390$$

The periodicity on this CD is 2 and its rate per (6-month) time period is 1.95%. The annualized rate of 3.90% assumes replication of the 6-month transaction on the very same terms.

Equation 1.4 can be used to obtain the ex-post rate of return realized by the money market mutual fund that purchased the CD and then sold it two months later to the dealer. Substitute in PV = \$1,000,000, FV = \$1,007,013, and $Days = 60$.

$$AOR = \left(\frac{360}{60}\right) * \left(\frac{\$1,007,013 - \$1,000,000}{\$1,000,000}\right) = 0.0421$$

The 2-month holding-period rate of return turns out to be 4.21%. Notice that in this series of calculations, the meanings of PV and FV change. In one case PV is the original principal on the CD, in another it is the market value at a later date. In one case FV is the redemption amount at maturity, in another it is the sale price prior to maturity. Nevertheless, PV is always the first cash flow and FV is the second.

The mutual fund buys a 6-month CD at 3.90%, sells it as a 4-month CD at 3.72%, and realizes a 2-month holding-period rate of return of 4.21%. This statement, while accurate, contains rates that are annualized for different periodicities. Here 3.90% has a periodicity of 2, 3.72% has a periodicity of 3, and 4.21% has a periodicity of 6. Comparing interest rates that have varying periodicities can be a problem but one that can be remedied with a conversion formula. But first we need to deal with another problem—money market discount rates.

Money Market Discount Rates

Treasury bills, commercial paper, and bankers acceptances in the U.S. are quoted on a discount rate (DR) basis. The price of the security is a discount from the face value.

$$PV = FV - \left[FV * DR * \frac{Days}{Year}\right] \tag{1.5}$$

Here, PV and FV are the two cash flows on the security; PV is the current price and FV is the amount paid at maturity. The term in brackets is the amount of the discount—it is the future (or face) value times the annual discount rate times the fraction of the year. Interest is not "added on" to the principal; instead it is included in the face value.

The pricing equation for discount rate instruments expressed more compactly is:

$$PV = FV * \left[1 - \left(DR * \frac{Days}{Year}\right)\right] \tag{1.6}$$

Suppose that the money manager buys the 180-day CP at a discount rate of 3.80%. The face value is \$1,000,000. Following market practice, the "amount" of a transaction is the face value (the *FV*) for instruments quoted on a discount rate basis. In contrast, the "amount" is the original principal (the *PV* at issuance) for money market securities quoted on an add-on rate basis. The purchase price for the CP is \$981,000.

$$PV = \$1{,}000{,}000 * \left[1 - \left(0.0380 * \frac{180}{360}\right)\right] = \$981{,}000$$

What is the realized rate of return on the CP, assuming the mutual fund holds it to maturity (and, of course, there is no default by the issuer)? We can substitute the two cash flows into equation 1.4 to get the result as a 360-day *AOR* so that it is comparable to the bank CD.

$$AOR = \left(\frac{360}{180}\right) * \left(\frac{\$1{,}000{,}000 - \$981{,}000}{\$981{,}000}\right) = 0.03874$$

Notice that the discount rate of 3.80% on the CP is a misleading growth rate for the investment—the realized rate of return is higher at 3.874%.

The rather bizarre nature of a money market discount rate is revealed by rearranging the pricing equation 1.6 to isolate the *DR* term.

$$DR = \left(\frac{Year}{Days}\right) * \left(\frac{FV - PV}{FV}\right) \tag{1.7}$$

Note that the *DR*, unlike an *AOR*, is not an APR because the second term in parentheses is not the periodic interest rate. It is the interest earned (*FV* – *PV*), divided by *FV*, and not by *PV*. This is not the way we think about an interest rate—the growth rate of an investment should be measured by the increase in value (*FV* – *PV*) given where we start (*PV*), not where we end

(FV). The key point is that discount rates on T-bills, commercial paper, and bankers acceptances in the U.S. systematically *understate* the investor's rate of return, as well as the borrower's cost of funds.

The relationship between a discount rate and an add-on rate can be derived algebraically by equating the pricing equations 1.3 and 1.6 and assuming that the two cash flows (PV and FV) are equivalent.

$$AOR = \frac{Year * DR}{Year - (Days * DR)} \tag{1.8}$$

The derivation is in the Technical Appendix. Notice that the AOR will always be greater than the DR for the same cash flows, the more so the greater the number of days in the time period and the higher the level of interest rates. Equation 1.8 is a general conversion formula between discount rates and add-on rates when quoted for the same assumed number of days in the year.

We can now convert the CP discount rate of 3.80% to an add-on rate assuming a 360-day year.

$$AOR = \frac{360 * 0.0380}{360 - (180 * 0.0380)} = 0.03874$$

This is the same result as given earlier—there the AOR equivalent is obtained from the two cash flows; here it is obtained using the conversion formula. If the risks on the CD and the CP are deemed to be equivalent, the money manager likes the CD. Doing the bond math, the manager expects a higher return on the CD because 3.90% is greater than 3.874%, not because 3.90% is greater than 3.80%. The key point is that add-on rates and discount rates cannot be directly compared—they first must be converted to a common basis. If the CD is perceived to entail somewhat more credit or liquidity risk, the investor's compensation for bearing that relative risk is only 2.6 basis points, not 10 basis points.

Despite their limitations as measures of rates of return (and costs of borrowed funds), discount rates are used in the U.S when T-bills, commercial paper, and bankers acceptances are traded. Assume the money market mutual fund manager has chosen to buy the $1,000,000, 180-day CP quoted at 3.80%, paying $981,000 at issuance. Now suppose that the manager seeks to sell the CP five months later when only 30 days remain until maturity, and at that time the securities dealer quotes a bid rate of 3.35% and an ask rate of 3.33% on 1-month CP. Those quotes will be on a discount rate basis. The dealer at that time would pay the mutual fund $997,208 for the security.

$$PV = \$1{,}000{,}000 * \left[1 - \left(0.0335 * \frac{30}{360}\right)\right] = \$997{,}208$$

How did the CP trade turn out for the investor? The 150-day holding period rate of return realized by the mutual fund can be calculated as a 360-day *AOR* based on the two cash flows:

$$AOR = \left(\frac{360}{150}\right) * \left(\frac{\$997{,}208 - \$981{,}000}{\$981{,}000}\right) = 0.03965$$

This rate of return, 3.965%, is an APR for a periodicity of 2.4. That is, it is the periodic rate for the 150-day time period (the second term in parentheses) annualized by multiplying by 360 divided by 150.

Two Cash Flows, Many Money Market Rates

Suppose that a money market security can be purchased on January 12th for $64,000. The security matures on March 12th, paying $65,000. To review the money market calculations seen so far, let's calculate the interest rate on the security to the nearest one-tenth of a basis point, given the following quotation methods and day-count conventions:

- Add-on Rate, Actual/360
- Add-on Rate, Actual/365
- Add-on Rate, 30/360
- Add-on Rate, Actual/370
- Discount Rate, Actual/360

Note first that interest rate calculations are *invariant to scale*. That means you will get the same answers if you simply use $64 and $65 for the two cash flows. However, if you work for a major financial institution and are used to dealing with large transactions, you can work with $64 million and $65 million to make the exercise seem more relevant. Interest rate calculations are also *invariant to currency*. These could be U.S. or Canadian dollars. If you prefer, you can designate the currencies to be the euro, British pound sterling, Japanese yen, Korean won, Mexican peso, or South African rand.

Add-on Rate, Actual/360

Actual/360 means that the fraction of the year is the actual number of days between settlement and maturity divided by 360. There are actually 59 days between January 12^{th} and March 12^{th} in non–leap years and 60 days during a leap year. A key word here is "between." The relevant time period in most financial markets is based on the number of days between the starting and ending dates. In other words, "parking lot rules" (whereby both the starting and ending dates count) do not apply.

Assume we are doing the calculation for 2011.

$$AOR = \left(\frac{360}{59}\right) * \left(\frac{\$65{,}000 - \$64{,}000}{\$64{,}000}\right) = 0.09534, \quad AOR = 9.534\%$$

Note that the periodicity for this add-on rate is 360/59, the reciprocal of the fraction of the year. If we do the calculation for 2012, the rate is a bit lower.

$$AOR = \left(\frac{360}{60}\right) * \left(\frac{\$65{,}000 - \$64{,}000}{\$64{,}000}\right) = 0.09375, \quad AOR = 9.375\%$$

Add-on Rate, Actual/365

Many money markets use actual/365 for the fraction of the year, in particular those markets that have followed British conventions. The add-on rates for 2011 and 2012 are:

$$AOR = \frac{365}{59} * \left(\frac{\$65{,}000 - \$64{,}000}{\$64{,}000}\right) = 0.09666, \quad AOR = 9.666\%$$

$$AOR = \frac{365}{60} * \left(\frac{\$65{,}000 - \$64{,}000}{\$64{,}000}\right) = 0.09505, \quad AOR = 9.505\%$$

In some markets, the number of days in the year switches to 366 for leap years. This day-count convention is known as actual/actual instead of actual/365.

The interest rate would be a little higher.

$$AOR = \frac{366}{60} * \left(\frac{\$65{,}000 - \$64{,}000}{\$64{,}000}\right) = 0.09531, \quad AOR = 9.531\%$$

Add-on Rate, 30/360

An easier way of counting the number of days between dates is to use the 30/360 day-count convention. Rather than work with an actual calendar (or use a computer), we simply assume that each month has 30 days. Therefore, there are *assumed* to be 30 days from January 12th to February 12th and another 30 days between February 12th and March 12th. That makes 60 days for the time period and 360 days for the year. We get the same rate for both 2011 and 2012:

$$AOR = \frac{360}{60} * \left(\frac{\$65{,}000 - \$64{,}000}{\$64{,}000}\right) = 0.09375, \quad AOR = 9.375\%$$

This day-count convention is rare in money markets but commonly is used for calculating the accrued interest on fixed-income bonds.

Add-on Rate, Actual/370

Okay, actual/370 does not really exist—but it could. After all, 370 days represents on average a year more accurately than does 360 days. Importantly, the calculated interest rate to the investor goes up. Assume 59 days in the time period.

$$AOR = \frac{370}{59} * \left(\frac{\$65{,}000 - \$64{,}000}{\$64{,}000}\right) = 0.09799, \quad AOR = 9.799\%$$

Think of the marketing possibilities for a commercial bank that uses 370 days in the year for quoting its deposit rates: "We give you five extra days in the year to earn interest!" Of course, the cash flows have not changed. The future cash flow (the *FV*) is the initial amount (the *PV*) multiplied by one plus the annual interest rate times the fraction of the year. For the same cash flows and number of days in the time period, raising the assumed number of days in the year lowers the fraction and "allows" the quoted annual interest rate to be higher. Why hasn't a bank thought of this?

Discount Rate, Actual/360

Discount rates by design always understate the investor's rate of return and the borrower's cost of funds. Assume again that the year is 2011.

$$DR = \frac{360}{59} * \left(\frac{\$65{,}000 - \$64{,}000}{\$65{,}000}\right) = 0.09387, \quad DR = 9.387\%$$

Note that this discount rate can be restated as an equivalent 360-day add-on rate using the conversion equation 1.8, matching the earlier result.

$$AOR = \frac{360 * 0.09387}{360 - (59 * 0.09387)} = 0.09534, \quad AOR = 9.534\%$$

It is critically important to know the rate quotation and day-count convention when working with money market interest rates. This example demonstrates that many different money market interest rates can be used to summarize the two cash flows on the transaction. It is also important to know when one rate needs to be converted for comparison to another. For example, to convert a money market rate quoted on an actual/360 add-on basis to a full-year or 365-day basis, simply multiply by 365/360. However, a rate quoted on a 30/360 basis already is stated for a full year. It is a mistake to gross it up by multiplying by 365/360.

A History Lesson on Money Market Certificates

One of the big problems facing U.S. commercial banks back in the 1970s was *disintermediation* caused by the Federal Reserve's Regulation Q. Reg Q limited the interest rates that banks could pay on their savings accounts and time deposits. The problem was that from time to time interest rates climbed above the Reg Q ceilings, usually because of increasing rates of inflation. Depositors naturally transferred their savings out of the banks and into money market mutual funds, which were not constrained by a rate ceiling.

The banks finally got regulatory relief. In June 1980, commercial banks were allowed to issue 6-month money market certificates (MMCs) that paid the 6-month T-bill auction rate *plus* 25 basis points. On Monday, August 25, 1980, the T-bill auction rate was 10.25%. Would an investor rather have put $50,000 into a T-bill that paid 10.25% or an MMC that paid 10.50%? Let's assume there was no difference in credit risk because the MMC was covered fully by government deposit insurance.

Obviously, the naive person (one who has not studied bond math) thought that 10.50% on the MMC was a better deal than 10.25% on the T-bill. What the commercial banks did not advertise was that their 10.50% was an *add-on rate* set by adding 25 basis points to the T-bill auction rate, which in turn was quoted on a *discount rate* basis. To make an apples-to-apples comparison, it is essential to convert the 10.25% discount rate to an add-on basis. Assume that the number of days was 182 and that both rates were for a 360-day year. Using the conversion formula 1.8, the equivalent add-on rate for the T-bill was 10.81%.

$$AOR = \frac{360 * 0.1025}{360 - (182 * 0.1025)} = 0.1081$$

The investor clearly should have chosen the T-bill. Not only was the rate of return significantly higher (10.81% compared to 10.50%), the interest income on the T-bill was exempt from state taxes while the MMC was taxed.

The Monetary Control Act of 1980 officially phased out Reg Q for traditional savings accounts over the following six years, but the constraint effectively was gone because of the newly authorized types of deposits, such as MMCs, which paid going market rates. Also, the T-bill auction rate back then was the weighted average of the accepted competitive bid rates submitted by securities dealers. Successful bidders paid different prices based on their own bid (discount) rates. That created a problem known as the "winner's curse"—those who bid more aggressively paid higher prices for the very same security. In 1998, the Treasury adopted a single-price auction for all maturities whereby all successful bidders pay the same price based on the highest accepted rate. You might not remember, but 1980 was a year of incredible, unprecedented swings in market rates. The 6-month T-bill auction rate was 15.70% on March 28th, down to 6.66% on June 20th, and back up to 15.42% on December 19th. That was some serious interest rate volatility!

Periodicity Conversions

A commonly used bond math technique is to convert an annual percentage rate from one periodicity to another. In the bond market, the need for this conversion arises when coupon interest cash flows have different payment frequencies. For example, interest payments on most fixed-income bonds are made semiannually, but on some the payments are quarterly or annually. Identifying relative value necessitates comparing yields for a common periodicity. In the money market, the need for the conversion arises when securities have different maturities. The 1-month, 3-month, and 6-month LIBOR have periodicities of about 12, 4, and 2, respectively, depending on the actual number of days in the time period.

The general periodicity conversion formula is shown in equation 1.9.

$$\left(1 + \frac{APR_x}{x}\right)^x = \left(1 + \frac{APR_y}{y}\right)^y \tag{1.9}$$

APR_x and APR_y are annual percentage rates for periodicities of x and y. Suppose that an interest rate is quoted at 5.25% for monthly compounding. Converted to a quarterly compounding basis, the new APR turns out to be 5.273%. This entails a periodicity conversion from $x = 12$ to $y = 4$ and solving for APR_4.

$$\left(1 + \frac{0.0525}{12}\right)^{12} = \left(1 + \frac{APR_4}{4}\right)^4, APR_4 = 0.05273$$

The key idea is that the total return at the end of the year is the same whether one receives 5.25% paid and compounded monthly (at that same monthly rate) or 5.273% paid and compounded quarterly (at that same quarterly rate).

Suppose that another APR is 5.30% for semiannual compounding. Converting that rate to a quarterly basis (from $x = 2$ to $y = 4$) gives a new APR of 5.265%:

$$\left(1 + \frac{0.0530}{2}\right)^{2} = \left(1 + \frac{APR_4}{4}\right)^4, APR_4 = 0.05265$$

The general rule is that converting an APR from more frequent to less frequent compounding per year (e.g., from a periodicity of 12 to 4) raises the annual interest rate (from 5.25% to 5.273%). Likewise, converting an APR from less to more frequent compounding (2 to 4) lowers the rate (5.30% to 5.265%). Put on a common periodicity, we see that 5.25% with monthly compounding offers a slightly higher return than 5.30% semiannually.

Another periodicity conversion you are likely to encounter is from an APR to an *effective annual rate* (EAR) basis, which implicitly assumes a periodicity of 1.

$$\left(1+\frac{APR_x}{x}\right)^x = 1 + EAR \tag{1.10}$$

For example, an APR of 5.25% having a periodicity of 12 converts to an EAR of 5.378% while the APR of 5.30% having a periodicity of 2 converts to 5.370%.

$$\left(1+\frac{0.0525}{12}\right)^{12} = 1 + EAR, \quad EAR = 0.05378$$

$$\left(1+\frac{0.0530}{2}\right)^{2} = 1 + EAR, \quad EAR = 0.05370$$

Some financial calculators have the APR to EAR conversion equation already programmed (note that "EFF" is sometimes used instead of "EAR"). The APR often is called a "nominal" interest rate in contrast to the "effective" rate. This is common in textbooks and in academic presentations. The idea is that the EAR represents the total return over a year, assuming replication and interest compounding at the same rate. The APR also assumes replication but merely adds up the rates per period and neglects the impact of compounding in obtaining the annualized rate of return.

An acronym used with U.S. commercial bank deposits is APY, standing for annual percentage yield. This is just another expression for the EAR. So, if the nominal rate on a 6-month bank deposit is quoted at 4.00%, its APY is displayed to be 4.04%. The higher the level of interest rates and the greater the periodicity of the nominal rate, the larger is the difference between an APR and its APY. If the APR on a 1-month bank deposit rate is 12.00%, its APY is 12.68%. It should be no surprise that banks like to display prominently the APY on time deposits and the APR on auto loans.

Treasury Bill Auction Results

In early July 2008, the U.S. Treasury auctioned off a series of T-bills. The official reported results for the auctions are shown in Table 1.1. Each T-bill

TABLE 1.1 T-Bill Auction Results

Term	Maturity Date	Discount Rate	Investment Rate	Price (per $100 in par value)
4 week	07-31-2008	1.850%	1.878%	99.856111
13 week	10-02-2008	1.900%	1.936%	99.519722
26 week	01-02-2009	2.135%	2.188%	98.914708
52 week	07-03-2009	2.295%	2.368%	97.679500

was issued on July 3, 2008, and uses an actual/360 day count. The Investment Rate is sometimes called the coupon or bond equivalent rate. The intent is to report to investors an interest rate for the security that is more meaningful than the discount rate and that allows a comparison to Treasury note and bond yields.

The given prices are straightforward applications of pricing on a discount rate basis using equation 1.6:

$$\text{4 week:} \quad PV = 100 * \left[1 - \left(0.01850 * \frac{28}{360}\right)\right] = 99.856111$$

$$\text{13 week:} \quad PV = 100 * \left[1 - \left(0.01900 * \frac{91}{360}\right)\right] = 99.519722$$

$$\text{26 week:} \quad PV = 100 * \left[1 - \left(0.02135 * \frac{183}{360}\right)\right] = 98.914708$$

$$\text{52 week:} \quad PV = 100 * \left[1 - \left(0.02295 * \frac{364}{360}\right)\right] = 97.679500$$

The 4-week, 13-week, 26-week, and 52-week T-bills almost always have 28, 91, 182, and 364 days to maturity, respectively. They typically are issued and settled on a Thursday and mature on a Thursday. The 26-week T-bill this time had 183 days in its time period because New Year's Day got in the way.

The Investment Rate (IR) for each T-bill can be calculated by working with the cash flows or with a conversion formula. First, use equation 1.4 for add-on rates, letting $Year = 365$.

$$4 \text{ week: } IR = \left(\frac{365}{28}\right) * \left(\frac{100 - 99.856111}{99.856111}\right) = 0.01878, \; IR = 1.878\%$$

$$13 \text{ week: } IR = \left(\frac{365}{91}\right) * \left(\frac{100 - 99.519722}{99.519722}\right) = 0.01936, \; IR = 1.936\%$$

$$26 \text{ week: } IR = \left(\frac{365}{183}\right) * \left(\frac{100 - 98.914708}{98.914708}\right) = 0.02188, \; IR = 2.188\%$$

$$52 \text{ week: } IR = \left(\frac{365}{364}\right) * \left(\frac{100 - 97.679500}{97.679500}\right) = 0.02382, \; IR = 2.382\%$$

The first three results confirm the reported Investment Rates; the fourth is *wrong*. The "official" APR—the one reported by the Treasury—on the 52-week T-bill is 2.368% while our calculation here is 2.382%. Quips like "close enough for government work" are not acceptable in bond math.

Before resolving this discrepancy, we can attempt to confirm the reported Investment Rates using a conversion formula similar to equation 1.8.

$$IR = \frac{365 * DR}{360 - (Days * DR)} \tag{1.11}$$

This directly converts a 360-day discount rate to a 365-day add-on rate.

$$4 \text{ week:} \quad IR = \frac{365 * 0.01850}{360 - (28 * 0.01850)} = 0.01878, \quad IR = 1.878\%$$

$$13 \text{ week:} \quad IR = \frac{365 * 0.01900}{360 - (91 * 0.01900)} = 0.01936, \quad IR = 1.936\%$$

$$26 \text{ week:} \quad IR = \frac{365 * 0.02135}{360 - (183 * 0.02135)} = 0.02188, \quad IR = 2.188\%$$

$$52 \text{ week:} \quad IR = \frac{365 * 0.02295}{360 - (364 * 0.02295)} = 0.02382, \quad IR = 2.382\%$$

Notice that identical results are obtained using either the cash flows or the conversion formula and that again we have the wrong Investment Rate for the 52-week T-bill.

The source of the discrepancy is that the U.S. Treasury uses a different method to calculate the Investment Rate when the time to maturity exceeds six months. The *IR* for the 52-week T-bill is based on this impressive formula.

$$IR = \frac{-\frac{2 * Days}{365} + 2 * \sqrt{\left(\frac{Days}{365}\right)^2 - \left(\frac{2 * Days}{365} - 1\right) * \left(1 - \frac{100}{PV}\right)}}{\frac{2 * Days}{365} - 1} \tag{1.12}$$

Enter $Days = 364$ and $PV = 97.679500$ to obtain the “correct” result that $IR = 2.368\%$ for the long-dated T-bill.

$$IR = \frac{-\frac{2 * 364}{365} + 2 * \sqrt{\left(\frac{364}{365}\right)^2 - \left(\frac{2 * 364}{365} - 1\right) * \left(1 - \frac{100}{97.679500}\right)}}{\frac{2 * 364}{365} - 1}$$

$$= 0.02368$$

Where does equation 1.12 come from? Mathematically, it is the solution to this expression found using the quadratic rule.

$$100 = PV * \left(1 + \frac{182.5}{365} * IR\right) * \left(1 + \frac{Days - 182.5}{365} * IR\right) \tag{1.13}$$

The equation is derived in the Technical Appendix. The Treasury’s intent is to provide an interest rate for the T-bill that is comparable to a Treasury note or bond that would mature on the same date and that still has one more coupon payment to be made.

A problem is that *IR* in equation 1.13 does not have a well-defined periodicity—and knowing the periodicity of an interest rate is critical in my opinion. The first term in parentheses in 1.13 looks like semiannual compounding for a periodicity of 2 (the annual rate of *IR* is divided by two periods in the year). The second term suggests compounding more frequently than semiannually. For example, if $Days = 270$, it looks like close to quarterly compounding (*IR* is divided by about four periods in the year). Frankly, the Investment Rates reported in financial markets on long-dated T-bills are not particularly transparent: knowing the rate and one cash flow does not allow one to calculate easily the other cash flow. Even discount rates, despite their inadequacy as rates of return, are transparent in that sense.

Suppose that we need to construct a Treasury yield curve. The idea of any yield curve in principle is to display visually the relationship between interest rates on securities that are alike on all dimensions except maturity. Ideally, all the observations would be for securities that have the same credit risk, same liquidity, and same tax status. That is why Treasury yield curves in the financial press typically are based on the most recently auctioned instruments (these are called the "on-the-run" securities). They not only are the most liquid, they also are priced close to par value. That mitigates tax effects due to prices at a premium or a discount to par value. That said, it is very common in practice to see the short end of the Treasury yield curve—that is, money market rates—display interest rates having varying periodicities.

Which T-bill rates should one include in a Treasury yield curve? Surely not the discount rates (1.850%, 1.900%, 2.135%, and 2.295%). Those understate the investor's rate of return. In my opinion, the best visual display of market conditions would report annual percentage rates having the same periodicity. A natural candidate is semiannual compounding because that is how yields to maturity on Treasury notes and bonds are calculated and presented.

Therefore, I suggest that T-bill discount rates first be converted to a 365-day add-on basis and then be converted to a *semiannual bond basis* (*SABB*). Note that $SABB = APR_2$ in equation 1.9—it is the APR for a periodicity of 2.

$$\text{4 week:} \quad \left(1 + \frac{0.01878}{365/28}\right)^{365/28} = \left(1 + \frac{SABB}{2}\right)^2, \quad SABB = 0.01886$$

$$\text{13 week:} \quad \left(1 + \frac{0.01936}{365/91}\right)^{365/91} = \left(1 + \frac{SABB}{2}\right)^2, \quad SABB = 0.01941$$

$$\text{26 week:} \quad \left(1 + \frac{0.02188}{365/183}\right)^{365/183} = \left(1 + \frac{SABB}{2}\right)^2, \quad SABB = 0.02188$$

$$\text{52 week:} \quad \left(1 + \frac{0.02382}{365/364}\right)^{365/364} = \left(1 + \frac{SABB}{2}\right)^2, \quad SABB = 0.02368$$

Each APR on the left side of each equation is the *IR* calculated above, including the "wrong" rate for the 52-week T-bill. The conversions of the 4-week and 13-week T-bills entail more frequent to less frequent compounding, so their

SABB rates are higher than the *IR*. The 26-week *SABB* is the same as the *IR* because 365/183 is so close to 2. Notice that the 52-week *SABB* is the same as the "correct" *IR* obtained with equation 1.12. That is because when $Days =$ 364, equation 1.13 effectively implies semiannual compounding.

Market practice, in any case, is to use the reported Investment Rates (1.878%, 1.936%, 2.188%, and 2.368%) at the short end of Treasury yield curves. This imparts a systematic bias for an upwardly sloping term structure because the shortest maturity rates have higher periodicities than the others. Best practice, I contend, would be to use the rates that have been converted to the *SABB* basis (1.886%, 1.941%, 2.188%, and 2.368%).

The differences between the *SABB* and the *IR* results in the example are quite small because the interest rates are low. Suppose instead that money market rates in the U.S. someday are much higher than they were in 2008. If the discount rates for each of these four T-bills are 12%, the "official" Investment Rates would be 12.281%, 12.547%, 12.957%, and 13.399%. Converted as above, the corresponding *SABB* rates would be 12.605%, 12.745%, 12.956%, and 13.400%. The difference at the short end of the yield curve then would be quite significant—32.4 basis points (12.605% minus 12.281%) for the 4-week bills and 19.8 basis points (12.745% minus 12.547%) for the 13-week bills.

The Future: Hourly Interest Rates?

Suppose that some time in the not-so-distant future the fastest-growing financial institution in the world is Bank 24/7/52. Its success owes to pioneering use of *hourly* interest rates for loans and deposits. Its (add-on) rates on short-term large time deposits (>$1,000,000) are shown in Table 1.2. The APR quoted by Bank 24/7/52 assumes a 364-day year. For instance, 3.4944% is calculated as 0.0004% * 24 * 7 * 52.

To see how hourly interest rates might work, suppose a corporation makes a 52-hour, $5,000,000 time deposit at Bank 24/7/52. The redemption

TABLE 1.2 Hourly Interest Rates

Time Period	Rate per Hour	APR
1–8 hours	0.0004%	3.4944%
9–24 hours	0.0005%	4.3680%
25–72 hours	0.0006%	5.2416%

amount on the deposit can be calculated using an hourly version of equation 1.3. The corporation will receive $5,001,560 when the deposit matures.

$$FV = \$5{,}000{,}000 * \left[1 + \left(0.052416 * \frac{52}{24 * 364}\right)\right] = \$5{,}001{,}560$$

The fraction of the year no longer is the number of days divided by the assumed number of days in the year; it becomes the number of hours for the transaction divided by the assumed number of hours in the year.

Now suppose that 30 hours after making the time deposit, the corporation has sudden need for liquidity. Bank 24/7/52's policy is to buy back time deposits as a service to its regular corporate customers. The redemption amount is fixed once the deposit is issued. The present value of the time deposit after 30 hours have passed and 22 hours remain is again based on equation 1.3 but now solving for *PV*.

$$\$5{,}001{,}560 = PV * \left[1 + \left(0.043680 * \frac{22}{24 * 364}\right)\right], \; PV = \$5{,}001{,}010$$

Assuming no change in the bank's rates, the corporate customer receives $5,001,010. Notice that this neglects the bank's bid-ask spread on money market transactions. In fact, Bank 24/7/52 likely would buy the deposit at a slightly higher rate (and lower price).

How did the corporation do on its short-term investment? The realized rate of return for its 30-hour holding period can be calculated with an hourly version of equation 1.4. That turns out to be 5.8822% on a 364-day add-on basis.

$$AOR = \left(\frac{364 * 24}{30}\right) * \left(\frac{\$5{,}001{,}010 - \$5{,}000{,}000}{\$5{,}000{,}000}\right) = 0.058822$$

Suppose that, for consistency, the money manager likes to convert all rates of return to a semiannual bond basis. Equation 1.9 can be used to convert that *AOR* to a *SABB*, but first one additional step is needed.

In general, interest rates should be put on a full-year, 365-day basis before carrying out the periodicity conversion. That is because a *SABB* having a periodicity of 2 implicitly assumes two *evenly spaced* periods in the 365-day year, each period having 182.5 days. (Notice that this assumption is implicit in equation 1.13.) So, first we need to convert 5.8822% to an add-on rate for 365 days in the year by multiplying by 365/364.

$$\left(\frac{365}{364}\right) * 5.8822\% = 5.8984\%$$

This rate is now an APR for 292 periods in the year: (365 * 24)/30 = 292. The holding-period rate of return converted to an *SABB* is 5.9856%.

$$\left(1 + \frac{0.058984}{292}\right)^{292} = \left(1 + \frac{SABB}{2}\right)^{2}, SABB = 0.059856$$

Conclusion

There are a number of factors that can account for the difference between any two money market interest rates. Usually the rate spread is explained by differences in credit risk, liquidity, taxation, and time to maturity. This chapter has emphasized more technical and mathematical factors, such as the method of rate quotation, the assumed number of days in the year, and the manner in which the rate per time period has been annualized. Many interest rates reasonably summarize the two cash flows on a money market security—and a significant subset of those many rates actually are used in practice.

Money market interest rates can be misleading and confusing to those who do not know the differences between add-on rates, discount rates, and interest rates in textbook time-value-of-money theory. Some rates are relics of an era when interest rate and cash flow calculations were made without computers and use arcane assumptions such as 360 days in the year. Knowing only the quoted interest rate on a money market security is not sufficient. You must also know its quotation basis, its day-count convention, and its periodicity. Only then do you have enough information to make a meaningful decision.