

Part I
BUILDING BLOCKS

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1

Interest Rate

In this chapter we review the idea of interest rate and the closely related concepts of compounding and discounting.

1-1 Measuring Time

In finance the standard unit of time is the year. But can we safely assume that a year has 365 days? What about the 366 days of a leap year? What fraction of a year does the first six months represent: 0.5, or $181/365$ (except, again, for leap years)?

Financial markets have regulations and conventions to answer these questions. The problem is that these conventions tend to vary by country. Worse still, within a given country different conventions may apply to different financial products.

We leave it to readers to become familiar with these day count conventions while in this book we will use the following rule, which professionals call **30/360** (Table 1-1 below). Note that the initial date starts at noon and the final date ends at noon; thus, there is only one whole day between 2 February 2012 and 3 February 2012.

Table 1-1 The 30/360 rule for measuring time

Rule	Result	Example: from 15 January 2012 to 13 March 2015
1. Count the number of whole years	Y	3 (from 15 January 2012 to 15 January 2015)
2. Count the number of remaining whole months and divide by 12	M/12	1/12 (from 15 January 2015 to 15 February 2015)
3. Count the number of remaining days (the last day of the month counting as the 30th unless it is the final date) and divide by 360	D/360	28/360 (under the 30/360 convention there are 16 days from 15 February 2015 to 1 March 2015 at noon and 12 days from 1 March 2015 to 13 March 2015)
TOTAL	Y + M/12 + D/360	3 + 1/12 + 28/360 = 3.161111...

From this rule we obtain the following simplified measures:

Semester (half year)	0.5 year
Quarter (three months)	0.25 year
Month	1/12 year
Week	7/360 year
Day	1/360 year

In practice...

The Excel function `DAYS360(Start_date, End_date)` counts the number of days on a 30/360 basis.

1-2 Interest Rate

In business life one can encounter two types of individuals whose *interests* are by definition opposed to each other:

- **Investors**, who have money and want to get richer while they remain idle;
- **Entrepreneurs**, who don't have money but want to become rich using the money of others.

Banks help to reconcile these two interests by acting as intermediaries, placing the money of the investor at the entrepreneur's disposal while taking the risk of bankruptcy (see Figure 1-1). In exchange, the bank demands that the entrepreneur pay *interest* at regular intervals, which serves to pay for the bank's service and the investor's capital.

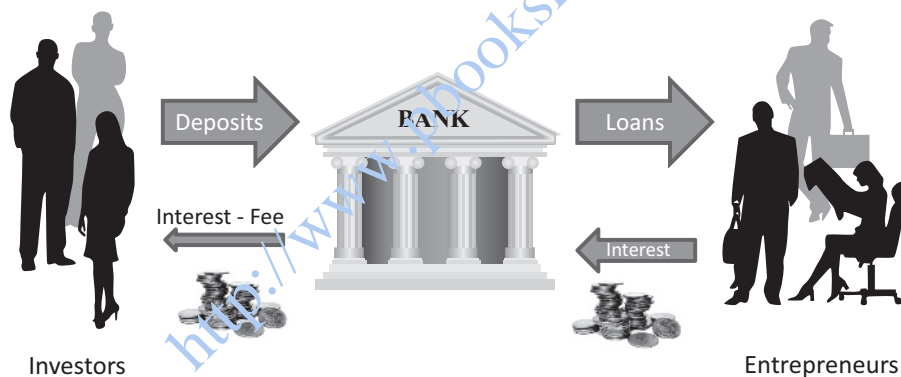


Figure 1-1 Banks are intermediaries between investors and entrepreneurs

1-2.1 Gross Interest Rate

Consider an investor who deposits \$100 and receives a total interest of \$12 over 2 years. His gross 2-year interest rate is then 12%. Generally, if I is the *total* interest paid on a capital K , the **gross interest rate** over the period in consideration is defined as:

$$r = \frac{I}{K}.$$

Examples

- €10 of interest paid over one year on a capital of €200 corresponds to a 5% annual gross interest rate.

- \$10 of interest paid *every year* for five years on a capital of \$200 corresponds to a 25% gross interest rate over five years, which is five times the above annual rate.

We must emphasize that **an interest rate is meaningless if no time period is specified**: a 5% gross interest rate every six months is far more lucrative than every year.

This rate is called ‘gross’ because it does not take into consideration the **compounding of interest**, which is explained next.

1-2.2 Compounding. Compound Interest Rate

When asked: “How much total interest does one collect after two years if the annual interest rate is 10%?” a distressing proportion of individuals reply in a single cry: “20%!” However, the correct answer is 21%, because **interest generates more interest**. In fact, a good capitalist, rather than foolishly spend the 10% interest paid by the bank after the first year, would immediately reinvest it the second year. Therefore, his total capital after one year is 110% of his initial investment on which he receives 10% interest the second year. His gross interest over the 2-year period is thus: $10\% + 10\% \times 110\% = 21\%$.

Generally, starting with initial capital K one may build a **compounding table** of capital at the end of each interest period (Table 1-2):

Table 1-2 Compounding table of capital K at interest rate r over n periods

Period	Capital	Example: $r = 10\%$
0	K	\$2,000
1	$K(1 + r)$	$2,000 \times (1 + 10\%) = \$2,200$
2	$K(1 + r)^2$	$2,200 \times (1 + 10\%) = \$2,420$
...
n	$K(1 + r)^n$	$2,000 \times (1 + 10\%)^n$

From this table we derive a formula for the amount of accumulated interest after n periods:

$$I_n = K(1 + r)^n - K.$$

We may now define the **compound interest rate** over n periods corresponding to the total accumulated interest:

$$r^{[n]} = \frac{I_n}{K} = (1 + r)^n - 1.$$

(To avoid confusion we prefer the notation $r^{[n]}$ over r_n to indicate compounding over n periods, as r_n typically denotes a series of time-dependent variables.)

Example

The total accumulated interest over 3 years on an initial investment of \$2,000 at 5% semi-annual gross interest rate is $I_6 = 2,000 \times (1 + 0.05)^6 - 2,000 = \680 . The compound interest rate over 3 years (6 semesters) is $r^{[6]} = 34\%$.

1-2.3 Conversion Formula

Two compound interest rates over periods τ_1 and τ_2 are said to be **equivalent** if they satisfy:

$$[1 + r^{[\tau_1]}]^{\frac{1}{\tau_1}} = [1 + r^{[\tau_2]}]^{\frac{1}{\tau_2}}. \tag{1-1}$$

Here τ_1 and τ_2 are measured in years (for instance $\tau_1 = 1.5$ represents a year and a half) and $r^{[\tau_1]}$ and $r^{[\tau_2]}$ are the equivalent interest rates over τ_1 and τ_2 years respectively.

Example

An investment at 5% semi-annual gross interest rate is equivalent to an investment at a 2-year compound rate of $r^{[2]} = (1 + 5\%)^{\frac{2}{0.5}} - 1 \approx 21.55\%$.

Equation (1-1) is very useful to convert a compound rate into a different period from the “physical” interest payment period. A good way to remember it is to think that for a given investment all expressions of the form $[1 + r^{[\text{period}]}]^{\text{frequency}}$ are equal, where frequency is the number of periods per year (the inverse of the period length).

1-2.4 Annualization

Annualization is the process of converting an interest rate into its annual equivalent. This allows one to quickly compare the profitability of investments whose interests are paid out over different periods.

In this book, unless mentioned otherwise, all interest rates are understood to be on an annual basis or annualized. With this convention, the compound interest rate over T years may always be written as:

$$r^{[T]} = (1 + r^{[\text{annual}]})^T - 1.$$

Example

The annualized rate equivalent to a 5% semi-annual gross rate is:

$$r^{[1]} = (1 + 5\%)^{\frac{1}{0.5}} - 1 = 1.05^2 - 1 = 10.25\%,$$

from which we obtain the 2-year compound rate found in the previous example:

$$r^{[2]} = (1 + 10.25\%)^2 - 1 \approx 21.55\%.$$

1-3 Discounting

'Time is money.' In finance, this principle of the businessman has a very precise meaning: **a dollar today is worth more than a dollar tomorrow**. Two main reasons may be put forward:

- **Interest:** one dollar today produces interest between today and tomorrow;
- **Inflation:** the increase in consumer prices implies that one dollar buys more today than tomorrow.

With this principle in mind the next step is to determine **the value today of a dollar tomorrow** – or generally the **present value** of an amount received or paid out in the future.

1-3.1 Present Value

If \$100 invested today at 5% annual interest is worth $\$100 \times (1 + 5\%)^4 \approx \121.55 in four years, how much is \$100 received in four years worth today? The answer is given by a simple cross-division: $\$100 / (1 + 5\%)^4 \approx \$100 \div 1.2155 \approx \82.27 . In other words \$82.27 invested today at 5% interest rate grows to \$100 in four years. This amount is called the present value of \$100 received in four years.

Generally, the **present value** PV of an amount C paid or received in T years is the equivalent amount which, invested today at rate r , grows to C : $PV \times (1 + r)^T = C$, i.e.:

$$PV = \frac{C}{(1 + r)^T}$$

Example

A European supermarket customarily pays its suppliers with a 3-month delay. With a 5% interest rate the present value of a delivery today of €1,000,000 worth of goods paid in 3 months is:

$$\frac{1,000,000}{(1 + 5\%)^{0.25}} \approx \text{€}987,877.$$

The 3-month payment delay is thus an implicit €12,123 discount, or 1.21%.

Discounting is the process of computing the present value of various future cash flows. It is a key concept in finance which brings all future amounts to their equivalent value as of today. Figure 1-2 below shows how discounting and compounding are reciprocal processes.

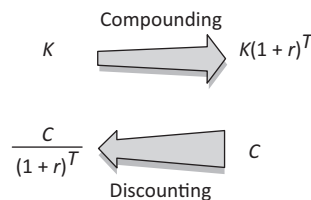


Figure 1-2 Compounding and discounting

1-3.2 Discount Rate and Required Return

In practice, the choice of the **discount rate** r is crucial when calculating a present value and depends on each investor's **required return**. The minimum required return is the interest rate offered by such "infallible" institutions as central banks or government treasury departments.

In the US, the generally accepted benchmark rate is the yield¹ of the 10-year Treasury Note. In Europe, the 10-year Gilt (UK), OAT (France), or Bund (Germany) are used, and in Japan the 10-year JGB.

However, an investor who is willing to take more risk should require a higher return and use a higher discount rate r . In investment banking it is not uncommon to use 10% to 20% discount rates when assessing the profitability of such risky investments as financing a film production or providing seed capital to a start-up company.

1-4 Problems

Problem 1: Measuring time

Calculate in years the time that passes between 30 November 2014 and 1 March 2016 on a 30/360 basis. What is the annualized interest rate of an investment at a gross rate of 10% over this period?

Problem 2: Savings account

On 1 January 2012 you deposit €1,000 in a savings account. On 1 January 2013 the bank sends a summary statement indicating that you received €40 of total interest in 2012.

- What is the gross annual interest rate of this savings account?
- How much interest will you get in 2013?
- Assuming interest is calculated and paid every month based on the account balance, how much interest would you have received if you had closed your account on 1 July 2012?

Problem 3: Ten years ago you invested £500 in a savings account. The last bank statement shows a balance of £1,030.52. What will your savings amount to in ten years if the interest rate stays the same?

Problem 4: From Greece with interest

You are a reputed financier and your personal credit allows you to borrow up to \$100,000 at a rate of 6.5% (with a little bit of imagination). On 22 September 2011 the interest offered on 1-year "deposits" with the Greek government is 135%, and the exchange rate of one euro is \$1.35. Your analysts believe that this exchange rate will remain stable during the coming year. Can you find a way to make money? Analyze your risks.

Problem 5: Sort the interest rates below from most lucrative to least lucrative:

- 6% per year;
- 0.5% per month;
- 30% every five years;
- 10% the first year then 4% the following two years.

¹ See Section 3-3 p.24 for the definition of yield.

Problem 6: Credit card

You receive a credit card offer with no minimum monthly payment and an annual interest rate of 17%, which is calculated and charged on your balance every month.

- (a) What is the compound interest rate charged by the credit card company if you pay off your balance after one month? Eighteen months? Five years?
- (b) Draw the curve of the compound interest rate as a function of time.
- (c) When will the amount of interest charged exceed the initial balance?

Problem 7*: Continuous interest rate

- (a) Consider the sequence $u_n = \left(1 + \frac{1}{n}\right)^n$ for $n \geq 1$. Show that (u_n) has limit e (Euler's constant: $e \approx 2.71828$.) *Hint: $x^y = e^{y \ln x}$, $\ln(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \dots$ for small h .*
- (b) If A_2 is a savings account with a 5% annual interest rate split into two payments of 2.5% every 6 months, what is the corresponding annualized interest rate r_2 ?
- (c) Let A_n be a savings account with an annual interest rate of 5% split into n payments. Find the corresponding annualized interest rate r_n as a function of n .
- (d) Find the limit r of r_n as n goes to infinity. How would you interpret the rate r ?

Problem 8: Discounting. Using a discount rate of 4% per annum, calculate the present value of:

- (a) €100,000 in one year?
- (b) €1,000,000 in ten years?
- (c) €100,000 ten years ago?

Problem 9: Expected return

After hesitating at length Mr Smith, an accomplished investment banker, eventually renounced an investment project offered at 30 million pounds sterling against a promised payoff of one billion pounds in twenty years' time. Can you estimate his required return?

Problem 10: Today's value of one dollar tomorrow

On 14 April 2005 the annualized interest rate on an overnight dollar deposit (i.e. between 14 April and 15 April 2005) was 2.77%. Calculate 'the value today of a dollar tomorrow,' that is the present value as of 14 April 2005 of one dollar collected on 15 April 2005.

Problem 11: Tuition planning

Mr and Mrs Jones are planning for their ten-year-old daughter Anna's future college education. According to College Board, a nonprofit membership association of more than 3,900 schools, colleges, and universities in the US, tuition at four-year private colleges and universities averaged \$27,293 per year in 2010–11. The Jones have \$20,000 in available savings and their bank offers a special deposit account with 5% guaranteed interest rate until Anna graduates from high school at the expected age of 18.

- (a) Assuming no inflation in tuition costs and a \$20,000 initial deposit, calculate how much the Joneses must save every year if they wish to cover for Anna's college education in full (room and board excepted)?

- (b) In 2000–01 college tuition averaged approximately \$20,308. Calculate the corresponding annualized inflation rate, and re-answer the previous question taking tuition inflation into account.

Problem 12*: Rule of 72

In 1494, the Italian mathematician Luca Pacioli wrote without proof: “In wanting to know for any percentage, in how many years the capital will be doubled, you bring to mind the rule of 72, which you always divide by the interest, and the result is in how many years it will be doubled. Example: When the interest is 6 percent per year, I say that one divides 72 by 6; obtaining 12, and in 12 years the capital will be doubled.” Show that this is approximately correct when the annual interest rate r is small enough. *Hint: $\ln(1 + h) \approx h$ for small h .*

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