Chapter 2
The Cheyette Model Class

The HJM framework is well-established in academia and practise to price and hedge interest rate derivatives. Imposing a special time dependent structure on the forward rate volatility function leads directly to the class of Cheyette models. In contrast to the general HJM model, the dynamics are Markovian, which allows the application of standard econometric valuation concepts. Finally, we distinguish this approach from alternative settings discussed in literature.

2.1 The Heath-Jarrow-Morton Framework

In 1992, Heath, Jarrow, and Morton (1992) standardized a valuation approach for interest rate derivatives on the basis of mainly two assumptions: the first one postulates, that it is not possible to gain riskless profit (No-arbitrage condition), and the second one assumes the completeness of the financial market. The Heath-Jarrow-Morton (HJM) model, or strictly speaking the HJM framework, is a general model environment and incorporates many previously developed approaches, such as Ho and Lee (1986), Vasicek (1977) and Hull and White (1990).

The general setting mainly suffers from two disadvantages: first of all the difficulty to apply the model in market practice and second, the extensive computational complexity caused by the high-dimensional stochastic process of the underlying. The first disadvantage was improved by the development of the LIBOR market model, which combines the general risk-neutral yield curve model with market standards. The second disadvantage can be improved by restricting the general HJM model to a subset of models with a specific parametrization of the volatility function. The resulting system of Stochastic Differential Equations (SDE) describing the yield curve dynamics, breaks down from a high-dimensional process into a low-dimensional structure of Markovian processes. This approach was developed by Cheyette (1994).

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The class of Cheyette interest rate models is a specialization of the general HJM framework, which is why we present the general setup of Heath, Jarrow, and Morton (1992) first. In the second step we limit the class of models to a specific process structure of the yield curve dynamic by imposing a parametrization of the forward rate volatility. This technique guides us to the representation of the class of Cheyette models.

The Heath-Jarrow-Morton approach yields a general framework for evaluating interest rate derivatives. The uncertainty in the economy is characterized by the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where Ω is the state space, $\mathscr{F}$ denotes a σ-algebra representing measurable events, and $\mathbb{P}$ is a probability measure. The uncertainty is resolved over time $[0, T]$ according to a filtration $\{\mathscr{F}_t\}$. The dynamic of the (instantaneous) forward rate is given by an Itô process

$$\mathrm{d} f(t,T) = \mu(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}\widetilde{W}(t). \tag{2.1}$$

It is assumed, that $\widetilde{W}(t)$ is an M-dimensional ($\mathbb{P}$-)Brownian motion and let the drift $\mu = (\mu(t,T))_{t\in[0,T]}$ and the volatility of the forward rate $\sigma = (\sigma(t,T))_{t\in[0,T]}$ be M-dimensional, progressively measurable, adapted stochastic processes satisfying certain conditions on the regularity as presented by Heath, Jarrow, and Morton (1992):

$$\int_0^T |\mu(s,T)|\mathrm{d}s < \infty \quad \mathbb{P}-a.s. \ \forall\ 0 \le T \le T^*, \tag{2.2}$$

$$\int_0^T \sigma_j^2(s,T)\mathrm{d}s < \infty \quad \mathbb{P}-a.s. \ \forall\ 0 \le T \le T^*, j = 1,\ldots,M, \tag{2.3}$$

$$\int_0^{T^*} |f(0,s)|\mathrm{d}s < \infty \quad \mathbb{P}-a.s., \tag{2.4}$$

$$\int_0^{T^*} \Big(\int_0^u |\mu(s,u)|\mathrm{d}s \Big) du < \infty \quad \mathbb{P}-a.s.. \tag{2.5}$$

The notations use the fixed maturities $T \ge t$ as well as the maximum time horizon $T^* \ge T$. Consequently, the forward rate results as

$$f(t,T) = f(0,T) + \int_0^t \mu(s,T)\mathrm{d}s + \int_0^t \sigma(s,T)\mathrm{d}\widetilde{W}(s). \tag{2.6}$$

The construction of the general forward rate is based on the assumption of a complete market. The economic concept of a complete market can be translated to the existence of a unique martingale measure as for example shown by Brigo and Mercurio (2006). This condition can, for instance, be fulfilled by assuming a restriction on the forward rate drift $\mu(t,T)$ as presented by Zagst (2002).

Lemma 2.1 (Forward Rate Drift Restriction). *If and only if the drift of the forward rate has the structure*

$$\mu(t,T) = \sigma(t,T)\left(\int_t^T \sigma(t,v)\,\mathrm{d}v - q(t)\right) \qquad \forall T \in [0,T^*] \text{ and } t \in [0,T],$$

where $q(t)$ denotes the Market Price of Risk*, then an equivalent martingale measure exists.*

The implied forward rate under the drift restriction is given by

$$f(t,T) = f(0,T) + \int_0^t \sigma(s,T)\left(\int_s^T \sigma(s,v)\mathrm{d}v - q(s)\right)ds + \int_0^t \sigma(s,T)\mathrm{d}\widetilde{W}(s). \tag{2.7}$$

Defining a different Wiener Process $W(t)$ generated by an equivalent martingale probability measure $\mathbb{Q}$ by

$$W(t) = \widetilde{W}(t) - \int_0^t q(s)\mathrm{d}s,$$

or in differential form by

$$\mathrm{d}W(t) = \mathrm{d}\widetilde{W}(t) - q(t)\mathrm{d}t,$$

one can eliminate the Market Price of Risk. Applying Girsanov's theorem[1] to (2.7), the forward rate results as

[1] The Radon-Nikodym derivative is given by

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \exp\Big(-\int_0^t \frac{1}{2}q^2(s)\mathrm{d}s - \int_0^t q(s)\mathrm{d}W(s)\Big).$$

We assume $\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^t \frac{1}{2}q^2(s)\mathrm{d}s - \int_0^t q(s)\mathrm{d}W(s)\Big)\Big] = 1$ to ensure that the Radon-Nikodyn derivative is a martingale.

$$f(t,T) = f(0,T) + \int_0^t \sigma(s,T) \left(\int_s^T \sigma(s,v)\mathrm{d}v \right) \mathrm{d}s + \int_0^t \sigma(s,T)\mathrm{d}W(s). \quad (2.8)$$

The stochastic integral equation can alternatively be expressed as a stochastic differential equation, thus the dynamic of the forward rate results as

$$\mathrm{d}f(t,T) = \sigma(t,T)\Big(\int_t^T \sigma(t,v)\mathrm{d}v\Big)\mathrm{d}t + \sigma(t,T)\mathrm{d}W(t). \quad (2.9)$$

The short rate $r(t)$ is the instantaneous spot rate, we can lock in at time t for borrowing at time t as for example defined by Shreve (2008). Consequently, the short rate is given by $r(t) = f(t,t)$. From (2.8) we conclude that the short rate can be expressed by the stochastic integral equation

$$r(t) = f(0,t) + \int_0^t \sigma(s,t)\Big(\int_s^t \sigma(s,v)\mathrm{d}v\Big)\mathrm{d}s + \int_0^t \sigma(s,t)\mathrm{d}W(s). \quad (2.10)$$

The instantaneous forward rate $f(t,T)$ at time t for date $T > t$ is often defined in relation to the bond price $B(t,T)$ by

$$f(t,T) = -\frac{\partial \ln B(t,T)}{\partial T},$$

or equivalently

$$B(t,T) = \exp\Big(-\int_t^T f(t,u)\mathrm{d}u\Big), \quad (2.11)$$

as for example done by Brigo and Mercurio (2006). Based on the representation of the forward rate (2.8), the dynamics of the bond price under the martingale measure $\mathbb{Q}$ result as

$$\mathrm{d}B(t,T) = B(t,T)\Big[r(t)\mathrm{d}t + b(t,T)\mathrm{d}W(t)\Big] \quad (2.12)$$

with bond price volatility

$$b(t,T) = -\int_t^T \sigma(t,u)\mathrm{d}u \quad (2.13)$$

as shown by Musiela and Rutkowski (2005).

Remark 2.2. If the volatility $\sigma(t,T)$ is a deterministic function, then the forward rate and the short rate are normally distributed.

The derivation of the forward rate dynamics is only based on the forward rate drift restriction, hence the HJM framework contains a really wide class of models. In this sense, the choice of a particular model depends uniquely on the specification of the forward rate volatility $\sigma(t,T)$. If the volatility function is deterministic, we end up in a Gaussian HJM model as indicated by Musiela and Rutkowski (2005).

2.2 Derivation of the Cheyette Model Class

The main problem in the HJM framework is the path dependency of the spot interest rate since it is non-Markovian in general. The non-Markovian structure of the dynamics makes it difficult to apply standard economic methods like Monte Carlo simulation or valuation via partial differential equations, because the entire history has to be carried, which increases the computational complexity and effort. The existing literature does not cover how to consistently express prices of contingent claims in terms of partial differential equations in the case of non-Markovian dynamics. Imposing a particular structure on the forward rate volatility it is possible to express the dynamics in terms of a finite dimensional Markovian system. In the following we will focus on the approach first published by Cheyette (1994).

Recall from Sect. 2.1, that in any arbitrage-free term structure model with continuous evolution of the yield curve, the dynamic of the forward rate is given by the stochastic differential equation (2.9). We can express this representation of the forward rate in the general M-Factor HJM model componentwise to highlight the multi-dimensionality in the form

$$df(t,T) = \sum_{k=1}^{M} \left[\sigma_k(t,T) \int_t^T \sigma_k(t,s)ds\right]dt + \sum_{k=1}^{M} \sigma_k(t,T)dW_k(t), \tag{2.14}$$

where $W(t)$ denotes an M-dimensional Brownian motion under the risk-neutral measure. The model is thus fully specified by a given volatility structure $\{\sigma(t,T)\}_{T\geq t}$ and the initial forward rate curve. The volatility function $\sigma(t,T)$ in an M-Factor model is an M-dimensional vector

$$\sigma(t,T) = \begin{pmatrix} \sigma_1(t,T) \\ \vdots \\ \sigma_M(t,T) \end{pmatrix}.$$

As already suggested in the literature, one can choose a specific volatility structure and achieve an exogenous model of the yield curve with Markovian dynamics. We follow the approach of Cheyette (1994) and use a separable volatility term structure. Each component $\sigma_k(t,T)$ is assumed to be separable into time and maturity dependent factors and is parameterized by a finite sum of separable functions, such that

$$\sigma_k(t,T) = \sum_{i=1}^{N_k} \frac{\alpha_i^{(k)}(T)}{\alpha_i^{(k)}(t)} \beta_i^{(k)}(t), \quad k = 1,\dots,M, \tag{2.15}$$

where N_k denotes the number of volatility summands of factor k. Different models with different characteristics can be obtained by the choice of the volatility function. If we assume the volatility structure (2.15), the forward rate can be reformulated as

$$f(t,T) = f(0,T) + \sum_{k=1}^{M} \left[\sum_{j=1}^{N_k} \frac{\alpha_j^{(k)}(T)}{\alpha_j^{(k)}(t)} \left(X_j^{(k)}(t) + \sum_{i=1}^{N_k} \frac{A_i^{(k)}(T) - A_i^{(k)}(t)}{\alpha_i^{(k)}(t)} V_{ij}^{(k)}(t) \right) \right] \tag{2.16}$$

with the state variables

$$X_i^{(k)}(t) = \int_0^t \frac{\alpha_i^{(k)}(t)}{\alpha_i^{(k)}(s)} \beta_i^{(k)}(s)\, \mathrm{d}W_k(s) + \int_0^t \frac{\alpha_i^{(k)}(t)\beta_i^{(k)}(s)}{\alpha_i^{(k)}(s)} \left[\sum_{j=1}^{N_k} \frac{A_j^{(k)}(t) - A_j^{(k)}(s)}{\alpha_j^{(k)}(s)} \beta_j^{(k)}(s) \right] \mathrm{d}s, \tag{2.17}$$

and the time functions

$$A_i^{(k)}(t) = \int_0^t \alpha_i^{(k)}(s)\mathrm{d}s, \tag{2.18}$$

$$V_{ij}^{(k)}(t) = V_{ji}^{(k)}(t) = \int_0^t \frac{\alpha_i^{(k)}(t)\alpha_j^{(k)}(t)}{\alpha_i^{(k)}(s)\alpha_j^{(k)}(s)} \beta_i^{(k)}(s)\beta_j^{(k)}(s)\mathrm{d}s, \tag{2.19}$$

for $k = 1,\dots,M$ and $i,j = 1,\dots,N_k$. The dynamics of the forward rate are determined by the state variables $X_i^{(k)}(t)$ for $k = 1,\dots,M$ and $i = 1,\dots,N_k$. Their dynamics are given by Markov processes as

$$\mathrm{d}X_i^{(k)}(t) = \left(X_i^{(k)}(t) \frac{\partial}{\partial t} \left(\log \alpha_i^{(k)}(t) \right) + \sum_{j=1}^{N_k} V_{ij}^{(k)}(t) \right) \mathrm{d}t + \beta_i^{(k)}(t)\mathrm{d}W_k(t). \tag{2.20}$$

Summarizing, the forward rate in an M-factor model is determined by $n = \sum_{k=1}^{M} N_k$ state variables, the $X_i^{(k)}(t)$. The representation of the forward rate (2.16) follows from (2.8) with the parametrization of the volatility (2.15). The details of these calculations and the derivation of the dynamics of the state variables $X_i^{(k)}(t)$ can be found in Appendix A.1.

The short rate $r(t)$ arises naturally as $r(t) = f(t,t)$ and applying the representation (2.16) leads directly to

$$r(t) = f(0,t) + \sum_{k=1}^{M} \sum_{j=1}^{N_k} X_j^{(k)}(t). \tag{2.21}$$

Consequently, the short rate is given as the sum of the initial forward rate and all state variables. Due to the parametrization of the volatility (2.15), the stochastic differential equation for $r(t)$ is Markovian.

2.3 Particular Models in the Cheyette Model Class

The model setup is quite general and in practise, it is popular to use parameterizations in the one-factor case ($M = 1$) of the form

$$\sigma(t,T) = \mathbb{P}_m(t) \exp\Big(-\lambda(T-t)\Big)$$

or

$$\sigma(t,T) = \mathbb{P}_m(t) \exp\Big(-\int_t^T \kappa(s)\mathrm{d}s\Big),$$

where $\mathbb{P}_m(t)$ denotes a polynomial of order $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ is a constant. These volatility parameterizations can be obtained in the class of Cheyette Models by choosing the volatility function (2.15) for $M = 1$ and $N_1 = 1$ as

$$\alpha_1^{(1)}(t) = \exp(-\lambda t), \quad \beta_1^{(1)} = \mathbb{P}_m(t)$$

or

$$\alpha_1^{(1)}(t) = \exp(-\int_0^t \kappa(s)\mathrm{d}s), \quad \beta_1^{(1)} = \mathbb{P}_m(t). \tag{2.22}$$

These parameterizations cover well-known models like the Ho-Lee Model, see Ho and Lee (1986), with

$$\sigma(t,T) = c$$

and the Hull-White Model, see Hull and White (1990), with

$$\sigma(t,T) = a \exp\Big(-\lambda(T-t)\Big).$$

2.3.1 Ho-Lee Model

The Ho-Lee Model is a one factor model ($M = 1$) and also a Gaussian HJM model with Markovian dynamics as shown by Ho and Lee (1986). The model is covered by the class of Cheyette models by choosing the volatility function as a constant

$$\sigma(t, T) = c.$$

In terms of the forward rate volatility parametrization (2.15), the volatility is expressed by

$$\alpha_1^{(1)}(t) = 1, \quad \beta_1^{(1)}(t) = c.$$

Thus the model is driven by one state variable $X_1^{(1)}(t)$. According to (2.17), the state variable results as

$$X_1^{(1)}(t) = \frac{1}{2}c^2t^2 + \int_0^t c\mathrm{d}W(s)$$

with dynamics, according to (2.20)

$$\mathrm{d}X_1^{(1)}(t) = tc^2\mathrm{d}t + c\mathrm{d}W(t). \tag{2.23}$$

Hence, the forward rate is given by

$$f(t, T) = f(0, T) + X_1^{(1)}(t) + (T - t)tc^2$$

and the short rate results as

$$r(t) = f(0, t) + X_1^{(1)}(t).$$

2.3.2 Hull-White Model

The Hull-White model or the extended Vasicek model of Hull-White is a one factor model ($M = 1$) with volatility

$$\sigma(t, T) = \tilde{c}\exp\Big(-\lambda(T - t)\Big)$$

for some constants $\tilde{c} \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. The model is part of the class of Cheyette models and can be obtained by choosing

$$\alpha_1^{(1)}(t) = \exp(-\lambda t), \quad \beta_1^{(1)}(t) = \tilde{c}$$

in (2.15). This leads directly to the representation of the only state variable according to (2.17)

$$X_1^{(1)}(t) = \frac{\tilde{c}^2}{2\lambda^2}\exp(-2\lambda t)\Big(-1+\exp(\lambda t)\Big)^2 + \tilde{c}\int_0^t \exp\Big(-\lambda(t-s)\Big)\mathrm{d}W(s).$$

The dynamic according to (2.20) results as

$$\mathrm{d}X_1^{(1)}(t) = \Big[-\lambda X_1^{(1)}(t) - \frac{\tilde{c}^2}{2\lambda}\Big(\exp(-2\lambda t)-1\Big)\Big]\mathrm{d}t + \tilde{c}\mathrm{d}W(t). \quad (2.24)$$

The forward rate results as

$$f(t,T) = f(0,T) + \exp\Big(-\lambda(T-t)\Big)\Big[X_1^{(1)}(t) - \frac{\Big(\exp(-\lambda T)-\exp(-\lambda t)\Big)\tilde{c}^2}{2\lambda^2\exp(-\lambda t)}\Big(\exp(-2\lambda t)-1\Big)\Big]$$

and the short rate is given by

$$r(t) = f(0,t) + X_1^{(1)}(t).$$

The Hull-White model contains three free parameters only and thus it doesn't provide enough flexibility to reproduce non-standard market states. The reproduction of a market state or the calibration of a model is an essential feature of an adequate model. However, the model can be extended easily by adding a constant term $c \in \mathbb{R}$, using linear polynomials and assuming $\kappa(s)$ to be a piecewise constant function in (2.22). Thus, the volatility function in the extended model is

$$\sigma(t,T) = c + (at+b)\exp\Big(-\int_t^T \kappa(s)\mathrm{d}s\Big). \quad (2.25)$$

The volatility function can be expressed in terms of the abstract volatility parametrization (2.15) by $K = 1$, $N_1 = 2$ and

$$\alpha_1^{(1)}(t) = 1, \qquad \beta_1^{(1)}(t) = c$$

$$\alpha_2^{(1)}(t) = at+b,\ \beta_2^{(1)}(t) = \exp\Big(-\int_0^t \kappa(s)\mathrm{d}s\Big). \quad (2.26)$$

The function $\kappa(s)$ is assumed to be piecewise constant and the domain is separated into two connected regions with constant function values κ_1 and κ_2. The number of partitions can be increased to improve the flexibility of the model. Assuming the volatility function (2.25), the one-factor model contains five free parameters. The calibration of Cheyette models, in particular the characterization of the optimization space, is analyzed in Chap. 4 based on this extension of the Hull-White model. Furthermore, we investigate the applicability and performance of optimization techniques in the calibration process. Therefore, we increase the flexibility of the model by improving the piecewise constant function $\kappa(s)$ and subdivide the domain into six connected regions with constant function values κ_i for $i = 1, \ldots, 6$. Consequently, the one-factor model contains nine free parameters.

2.3.3 The Three Factor Exponential Model

The class of Cheyette models as well as the HJM framework support a canonical extension of the models to incorporate several factors. Using multiple factors increases the power of the model, in the sense that it is more flexible and replicates market movements more realistically. First, the calibration of the model to a market state becomes more accurate and second, the simulation of forward rates gets more realistic. Thus, it is more adequate to be used in practice. Empirical studies, as for example presented by Cheyette (1994), have shown that it is sufficient to incorporate three factors in order to obtain reliable results. Therefore, we choose the three factor model ($M = 3$) with volatility

$$\sigma(t,T) = \begin{pmatrix} \sigma_1(t,T) \\ \sigma_2(t,T) \\ \sigma_3(t,T) \end{pmatrix}, \tag{2.27}$$

with

$$\sigma_1(t,T) = c + \mathbb{P}_m^{(1)} \exp\Big(-\lambda^{(1)}(T-t) \Big), \tag{2.28}$$

$$\sigma_2(t,T) = \mathbb{P}_m^{(2)} \exp\Big(-\lambda^{(2)}(T-t) \Big), \tag{2.29}$$

$$\sigma_3(t,T) = \mathbb{P}_m^{(3)} \exp\Big(-\lambda^{(3)}(T-t) \Big), \tag{2.30}$$

where $c \in \mathbb{R}$ and $\lambda^{(k)} \in \mathbb{R}$ are constants and $\mathbb{P}_m^{(k)} = a_m^{(k)} t^m + \ldots + a_1^{(k)} t + a_0^{(k)}$ denotes a polynomial of order m. Using this parametrization, one receives $N_1 = 2$, $N_2 = 1$ and $N_3 = 1$ in (2.15). Consequently, the Three Factor Exponential Model is driven by four state variables. The first factor is linked to the state variables $X_1^{(1)}(t)$ and $X_2^{(1)}(t)$ and the corresponding volatility parametrization is given by

$$\alpha_1^{(1)}(t) = 1, \quad \beta_1^{(1)}(t) = c,$$
$$\alpha_2^{(1)}(t) = \exp\Big(-\lambda^{(1)}t\Big), \quad \beta_2^{(1)}(t) = \mathbb{P}_m^{(1)}(t).$$

The second and third factors are associated with the state variables $X_1^{(2)}(t)$ and $X_1^{(3)}(t)$ respectively and the corresponding volatility parameterizations are given by

$$\alpha_1^{(2)}(t) = \exp\Big(-\lambda^{(2)}t\Big), \quad \beta_1^{(2)}(t) = \mathbb{P}_m^{(2)}(t),$$
$$\alpha_1^{(3)}(t) = \exp\Big(-\lambda^{(3)}t\Big), \quad \beta_1^{(3)}(t) = \mathbb{P}_m^{(3)}(t).$$

The parametrization of the volatility can be set up with arbitrary polynomial order, which influences the flexibility of the model. We use linear polynomials ($m = 1$) as an example and it follows that $\beta_2^{(1)}(t)$, $\beta_1^{(2)}(t)$, $\beta_1^{(3)}(t)$ reduce to $\mathbb{P}_1^{(k)}(t) = a_1^{(k)}t + a_o^{(k)}$ for $k = 1, 2, 3$.

We incorporate a constant term in the volatility function of the first factor to increase the accuracy of the model calibration, since the volatility tends to the constant term in the limit $T \to \infty$. Due to the additional constant term, the volatility of the first factor is shifted by a constant to a reasonable level. In particular, the calibration results for short maturities improve. The methodology and the results of the calibration are presented in Chap. 4.

The Three Factor Exponential Model is represented by four state variables whose dynamics are given by (2.17). In the following we will show the specific form in the Three Factor Exponential Model. The first factor is determined by the first component of the volatility, $\sigma_1(t, T)$, as specified by (2.28). The state variables $X_1^{(1)}(t)$ and $X_2^{(1)}(t)$ are associated with the first factor and their dynamics result as

$$dX_1^{(1)}(t) = \left(\sum_{k=1}^{2} V_{1k}^{(1)}(t)\right)dt + c\,dW_1(t), \tag{2.31}$$

$$dX_2^{(1)}(t) = \left(-\lambda^{(1)}X_2(t) + \sum_{k=1}^{2} V_{2k}^{(1)}(t)\right)dt + \left(a_1^{(1)}t + a_0^{(1)}\right)dW_1(t). \tag{2.32}$$

The dynamics of the state variables and the representation of the forward rate (2.16) are based on time-dependent functions $V_{ij}^{(k)}(t)$ defined in (2.19). In the Three Factor Exponential Model, these functions are linear combinations of exponential and quadratic functions of the volatility parameters and time. In particular, the functions are deterministic and can be calculated explicitly, see Appendix A.2.1.

The state variables of the second and third factor are associated with the second and third component of the volatility, $\sigma_2(t, T)$ and $\sigma_3(t, T)$, specified in (2.29) and (2.30). Their dynamics are given by

$$\mathrm{d}X_1^{(2)}(t) = \Big[-\lambda^{(2)} X_1^{(2)}(t) + V_{11}^{(2)}(t) \Big]\mathrm{d}t + \Big(a_1^{(2)} t + a_0^{(2)} \Big)\mathrm{d}W_2(t), \tag{2.33}$$

$$\mathrm{d}X_1^{(3)}(t) = \Big[-\lambda^{(3)} X_1^{(3)}(t) + V_{11}^{(3)}(t) \Big]\mathrm{d}t + \Big(a_1^{(3)} t + a_0^{(3)} \Big)\mathrm{d}W_3(t). \tag{2.34}$$

Again, the time dependent functions $V_{ij}^{(k)}(t)$ are deterministic and given in Appendix A.2.1. Later in the book, we sometimes use the following representation of the state variable $X_1 = X_1^{(1)}$, $X_2 = X_2^{(1)}$, $X_3 = X_1^{(2)}$ and $X_4 = X_1^{(3)}$ to improve the readability.

Remark 2.3. The Three Factor Exponential Model serves as a reference model throughout this book for pricing interest rate derivatives by Monte Carlo methods (Chap. 5), characteristic functions (Chap. 6) and PDE methods (Chap. 7). Therefore we have to calibrate the model and fix the parameter values of the volatility function (2.27). This is done in Chap. 4 for two different market states. Furthermore we derive analytical pricing formulas for bonds and caplets in the Three Factor Exponential Model which will serve as a benchmark to the numerical pricing methods. Finally, we develop formulas for risk sensitivities (Greeks) and apply them to plain-vanilla and exotic products in the Three Factor Exponential Model.

2.4 Remarks on the Cheyette Model Class

The class of Cheyette interest rate models has its origin in the general HJM framework and is specified by imposing a structure for the volatility function $\sigma(t,T)$. The approach of Cheyette assumes the volatility to be separable into time and maturity dependant functions according to (2.15). Thereby, it differs from the approach treated by Björk and Svensson (2001) and Chiarella and Bhar (1997). Björk showed in his work that a necessary condition for the existence of a Markovian realization of the HJM model is that the volatility $\sigma(T-t)$, as a function of time to maturity $T-t$ only, has the structure $\sigma(T-t) = \mathbb{P}_m(T-t)\exp\Big[-\lambda(T-t)\Big]$, where $\mathbb{P}_m(x)$ denotes a polynomial of order $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ is a constant. This approach allows deterministic volatility functions

$$\sigma(T-t) = \mathbb{P}_m(T-t)\exp(-\lambda(T-t))$$

as investigated by Chiarella and Bhar (1997) and can even be extended to

$$\sigma(T-t) = \mathbb{P}_m(T-t)\exp(-\lambda(T-t))G(r(t)),$$

where $G(r(t))$ denotes a function of the instantaneous spot rate of interest. Nevertheless, the volatility structure is dependant on the time to maturity only.

Cheyette (1994) showed, that one can achieve a Markovian realization of the HJM Model even if time and maturity are incorporated separately. Thus, it is possible to assume

$$\sigma(t,T) = \mathbb{P}_m(t)\exp\Big(-\lambda(T-t)\Big).$$

Furthermore, the structure can be extended arbitrarily as long as it can be written in the form of (2.15).

The order of the polynomial $\mathbb{P}_m(t)$ does not influence the number of state variables in the Cheyette class of models. Thus, one could choose an arbitrary polynomial order without boosting the complexity of the model. A one-factor model with one volatility summand incorporating a polynomial of order m only requires one state variable. In contrast, if we assume that the polynomial is dependent on the time to maturity $T-t$, as done by Björk and Svensson (2001) and Chiarella and Bhar (1997), the order of the polynomial does influence the number of state variables. For instance, a one-factor model incorporating a polynomial of order m requires $m+1$ state variables in the setup of Björk and Chiarella. More details can be found in Appendix A.5.

Summarizing, we would like to mention, that the presented framework is not equivalent to the work of Björk or Chiarella and has its origin in the work of Cheyette (1994), although there is a large overlap between the two approaches.

The HJM framework is very powerful and can be used to value all types of interest rate derivatives. Nowadays, these models are also used to model commodity derivatives as for example shown by Schwartz (1997) or Miltersen and Schwartz (1998). In particular, one often uses Gaussian HJM models derived by a deterministic volatility function. Therefore, it would be possible and promising to use the class of Cheyette models for pricing commodity derivatives.

The particular models in the class of Cheyette Models are determined by the choice of the volatility function, thus the characteristics and the accuracy depend on this choice. Increasing the complexity of the volatility function increases the power of the resulting model. It is possible to extend the volatility function to be dependent on the short rate or the forward rate. Furthermore one could assume a stochastic volatility. All extensions increase the accuracy, but they boost the complexity of the model as well. We restrict the volatility function to the form of (2.15) throughout this work.